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Boulder Laboratories

# MODE CALCULATIONS FOR <br> VLF PROPAGATION IN THE 

EARTH-IONOSPHERE WAVEGUIDE

BY<br>KENNETH P. SPIES AND JAMES R. WAIT

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Kenneth P. Spies and James R. Wait


#### Abstract

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# MODE CALCULATIONS FOR VLF PROPAGATION IN THE EARTH-IONOSPHERE WAVEGUIDE 

## by

Kenneth P. Spies and James R. Wait

## I. Introduction

The concept that radio waves are channeled between the earth and the ionosphere as in a waveguide has been very useful at VLF [Budden, 1953; Alpert, 1956; Wait, 1957]. Unfortunately, the computational aspects of the problem are quite complicated even when the model is highly idealized. The difficulty stems from the grazing nature of the modes of lowest attenuation. Some progress has been made recently by utilizing higher order approximations for the various spherical wave functions which enter into the problem. In this way the influence of earth curvature has been fully accounted for. The detailed theoretical aspects and essential derivations have been presented elsewhere [Wait, 1960, 1961]. Here the actual computational procedure is outlined and some numerical results are presented. It is believed that the methodsused are of general interest and also have possible application to propagation of acoustic and seismic waves in curved layered media.

An essential feature of the techniques used is a simplified representation of the ionospheric reflection coefficient which is valid for highly oblique incidence. This permits a perturbation procedure to be applied to the equation for the mode characteristics. In the examples employed here the ionosphere is represented by a sharply bounded and homogeneous ionized medium with a superimposed magnetic field. Initially the quasi-longitudinal approximation is
invoked since it permits a great simplification in the analysis. Its validity has been discussed in detail elsewhere [Budden, 1961]. Generally speaking, it is good when the propagation is in the magnetic meridian or in polar regions for all directions of propagation. These results are then introduced into the mode equation which involves the reflection coefficients $\|_{\|}$and ${ }_{\perp} R_{\perp}$ and the conversion coefficients $\|_{\perp}{ }^{\text {and }}{ }_{\perp} R_{\|}$evaluated for a complex angle of incidence. Here the earth is regarded as flat in order to simplify the calculation. In the following section the earth's magnetic field is taken to be horizontal and transverse to the direction of propagation. This would correspond to propagation along the magnetic equator. Then the case of an arbitrarily dipping magnetic field is treated in a relatively crude fashion. Most of the calculations mentioned above are then carried out for a curved earth where the modal equation is a great deal more complicated. Nevertheless, by making certain convergent expansions of the Airy functions, the problem becomes tractable and can be treated by perturbation methods.

For the convenience of those engaged in related investigations, a table of inverse tangents of the ratio of two Airy functions is given in Appendix A. A fairly extensive collection of formulas relating to Airy functions is then given in Appendix B. Finally, for sake of completeness, a short note on the concept of conductivity of an ionized gas is contained in Appendix C.

## II. Calculations for a Flat Earth-Ionosphere Waveguide

## 1. Ionospheric Reflection Coefficients in the Quasi-longitudinal

(Q-L) Approximation. - In the quasi-longitudinal (Q-L) approximation, ionospheric reflection and conversion coefficients for a sharply bounded homogeneous ionosphere have been given by Budder [1961] in the follow ing form

$$
\begin{align*}
& \left\|^{R}\right\|=\frac{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}-C_{o} C_{e}\right)+\left(\mu_{o} \mu_{e}-1\right)\left(C_{o}+C_{e}\right) C}{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}+C_{o} C_{e}\right)+\left(\mu_{o} \mu_{e}+1\right)\left(C_{o}+C_{e}\right) C},  \tag{1}\\
& \perp^{R_{\perp}}=\frac{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}-C_{o} C_{e}\right)-\left(\mu_{o} \mu_{e}-1\right)\left(C_{o}+C_{e}\right) C}{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}+C_{o} C_{e}\right)+\left(\mu_{o} \mu_{e}+1\right)\left(C_{o}+C_{e}\right) C},  \tag{2}\\
& \|^{R} \perp_{\perp}=\frac{\operatorname{i2C}\left(\mu_{o} C_{o}-\mu_{e} C_{e}\right)}{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}+C_{o} C_{e}\right)+\left(\mu_{o} \mu_{e}+1\right)\left(C_{o}+C_{e}\right) C},  \tag{3}\\
& \perp^{R} \|=\frac{i 2 C\left(\mu_{o} C_{e}-\mu_{e} C_{o}\right)}{\left(\mu_{o}+\mu_{e}\right)\left(C^{2}+C_{o} C_{e}\right)+\left(\mu_{o} \mu_{e}+1\right)\left(C_{o}+C_{e}\right) C}, \tag{4}
\end{align*}
$$

and
where $C$ is the cosine of the (complex) angle of incidence

$$
1-C^{2}=\mu_{0}^{2}\left(1-C_{0}^{2}\right)
$$

and where

$$
1-C^{2}=\mu_{e}^{2}\left(1-C_{e}^{2}\right) \quad \text { (Snell's Law) }
$$

The refractive indices $\mu_{o}$ and $\mu_{e}$ of the ionosphere are defined by the relations

$$
\mu_{o}^{2}=1-\frac{i}{L} e^{i \phi} L \text { and } \mu_{e}^{2}=1-\frac{1}{L} e^{-i \phi_{L}}
$$

where $\tan \phi_{L}=\frac{\omega_{L}}{\nu}=\frac{\text { longitudinal gyro frequency }}{\text { collisional frequency }}$,
where

$$
L=\frac{\omega}{\omega_{r}}=\frac{\omega\left(v^{2}+\omega_{L}^{2}\right)^{\frac{1}{2}}}{\omega_{\mathrm{O}}^{2}}
$$

and where $\omega$ is the angular frequency and $\omega_{r}$ is an effective conductivity [Wait, 1960] which is defined explicitly above. Note that ${ }_{\|} R_{\|}$and
 $\|_{\perp} R_{\perp}$ and $R_{\|}$both approach 0 .

To facilitate solving the made equation for VLF propagation in a flat earth-ionosphere waveguide, it is convenient to represent the rather complicated reflection coefficients $\left\|_{\|}\right\|^{\text {and }}{ }_{\perp} R_{\perp}$ by expressions having the form

$$
\begin{equation*}
R=-\exp \left[a_{1} C+a_{2} C^{2}+a_{3} C^{3}+\ldots\right] \tag{5}
\end{equation*}
$$

where the (complex) parameters $a_{1}, a_{2}, a_{3}, \ldots$ depend on ionospheric properties, but not on $C$. The validity of such a procedure depends on the fact that $|C|$ is small for the most important modes at VLF, so that $R$ is near -1 .

There is, of course, no unique way of choosing $a_{1}, a_{2}, a_{3}, \ldots$. The method finally adopted is the following. In

$$
\|_{\|} R_{\|}=e^{\log \|_{\|} R_{\|}} \text {and }{ }_{\perp} R_{\perp}=e^{\log } \perp_{\perp} R_{\perp},
$$

the logarithms are expanded in a Taylor series about $C=0$ to obtain

$$
\begin{equation*}
\left\|^{R}\right\|_{\|}=\exp \left[\sum_{k=0}^{\infty} a_{k}^{\infty} C^{k}\right] \text { and }{ }_{\perp} R_{\perp}=\exp \left[\sum_{k=0}^{\infty} \beta_{k} C^{k}\right] \tag{6a}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{k}=\frac{1}{k!}\left[\frac{\partial^{k}}{\partial C^{k}}\left(\log _{\|} R_{\|}\right)\right]_{C=0} \text { and } \beta_{k}=\frac{1}{k!}\left[\frac{\partial^{k}}{\partial C^{k}}\left(\log _{\perp} R_{\perp}\right)\right]_{C=0} . \tag{7a}
\end{equation*}
$$

The first few Taylor series coefficients for $\log \|_{\|} \mathrm{R}_{\|}$are

$$
\begin{equation*}
a_{0}=\mathrm{i} \pi \tag{8a}
\end{equation*}
$$

$$
\begin{equation*}
a_{1}=-\frac{2 \mu_{o} \mu_{e}}{\mu_{0}+\mu_{e}}\left[\frac{\mu_{e}}{\sqrt{\mu_{e}^{2}-1}}+\frac{\mu_{o}}{\sqrt{\mu_{o}^{2}-1}}\right] \tag{8b}
\end{equation*}
$$

$$
a_{2}=\frac{a_{1}^{2}}{2 \mu_{\mathrm{o}} \mu_{\mathrm{e}}}-\frac{2 \mu_{\mathrm{o}} \mu_{e}}{\sqrt{\mu_{\mathrm{o}}^{2}-1} \sqrt{\mu_{e}^{2}-1}}
$$

$a_{3}=\frac{a_{1}}{24}\left[a_{1}^{2}+\frac{6 a_{2}}{\mu_{o} \mu_{\dot{e}}}-\frac{6}{\mu_{e}^{2}-1}\right]+\frac{1}{2} \frac{\mu_{o} \mu_{e}}{\mu_{o}+\mu_{e}} \frac{\mu_{o}}{\sqrt{\mu_{o}^{2}-1}}\left[\frac{1}{\mu_{o}^{2}-1}-\frac{1}{\mu_{e}^{2}-1}\right]$,
and those for $\log _{\perp} R_{\perp}$ are

$$
\begin{gather*}
\beta_{o}=i \pi  \tag{9a}\\
\beta_{1}=\frac{a_{1}}{\mu_{\mathrm{o}} \mu_{\mathrm{e}}},  \tag{9b}\\
\beta_{2}=a_{2},  \tag{9c}\\
\beta_{3}=\frac{\beta_{1}}{24}\left(\mu_{\mathrm{o}}^{2} \mu_{\mathrm{e}}^{2}-1\right)\left(\frac{6 \beta_{2}}{\mu_{\mathrm{o}} \mu_{\mathrm{e}}}-\beta_{1}^{2}\right)+\frac{a_{3}}{\mu_{\mathrm{o}}^{\mu} \mu_{e}} \tag{9d}
\end{gather*}
$$

The ionospheric reflection coefficients $\left\|_{\|}\right\|_{\|}$and ${ }_{\perp} R_{\perp}$ can now be approximated by using a few terms of the series (6a) and (6b), respectively。
2. An Approximate Solution of the Mode Equation for Perfectly Conducting Ground - When the ground is perfectly conducting, the mode equation for VLF propagation in a flat earth-ionosphere waveguide is given by [Wait, 1960]

$$
\begin{equation*}
\left(e^{i 2 k h C}-\|_{\|}\right)\left(e^{i 2 k h C}+{ }_{\perp} R_{\perp}\right)+\|_{\perp} R_{\perp} R_{\|}=0 \tag{10}
\end{equation*}
$$

As a first approximation, coupling between the modes is neglected and thus ${ }_{\|} R_{\perp}={ }_{\perp} R_{\|}=0$. The mode equation (10) separates into the two equations

$$
\begin{array}{ll}
e^{i 2 k h C}-{ }_{\|} R_{\|}=0 & \text { (for } T M \text { modes) } \\
e^{i 2 k h C}+{ }_{\perp} R_{\perp}=0 & \text { (for TE modes) } \tag{12}
\end{array}
$$

If the ionospheric reflection coefficient $\|_{\|} \mathrm{R}_{\|}$is approximated by using the first two terms of the Taylor series (6a) for $\log { }_{\|} R_{\|}$, one has

$$
\|_{\|} R_{\|}=-e^{a_{1} C}
$$

where $a_{1}$ is given $b y(8 \mathrm{~b})$. With this approximation, the solutions of the TM mode equation (ll) are

$$
\begin{equation*}
C_{n}=\frac{(2 n-1) \pi}{2 k h+i a_{1}} \quad(n=1,2,3, \ldots), S_{n}=\sqrt{1-C_{n}^{2}} \tag{13}
\end{equation*}
$$

Curves of $1 / \operatorname{Re}\left(S_{n}\right)$ and $-H \operatorname{Im}\left(S_{n}\right)$ vs. $H=\frac{k h}{2 \pi} \quad$ have been computed when $n=1$ for
$B=\frac{1}{H}\left(\frac{\omega}{\omega_{r}}\right)=0.02,0.05,0.10,0.20$ and $\phi_{L}=0^{\circ}, 10^{\circ}, 20^{\circ}, 30^{\circ}, 40^{\circ}$,
$50^{\circ}, 60^{\circ}$. These results are shown in figures 1 through 11. (Only curves for $\phi_{L}=0^{\circ}$ and $\phi_{L}=60^{\circ}$ have been plotted.)

Likewise, if the ionospheric reflection coefficient ${ }_{\perp} R_{\perp}$ is approximated by using the first two terms of the Taylor series (6b) for $\log _{\perp} R_{\perp}$, one has

$$
\perp_{\perp} R_{\perp}=-e^{\beta_{I} C}
$$

where $\beta_{1}$ is given by (9b). With this approximation, the solutions of the TE mode equation (12) are

$$
\begin{equation*}
C_{m}=\frac{2 m \pi}{2 k h+i \beta_{1}}(m=1,2,3, \ldots) \tag{14}
\end{equation*}
$$

## FREQUENCY, kc/s



Fig. 1

FREQUENCY, kc/s


Fig. 2

FREQUENCY, $\mathrm{kc} / \mathrm{s}$


Fig. 3

FREQUENCY, kc/s


Fig. 4

## FREQUENCY, kc/s



Fig. 5

## FREQUENCY, kc/s



Fig. 6

FREQUENCY, kc/s


Fig. 7

FREQUENCY, kc/s


Fig. 8

## FREQUENCY, kc/s



Fig. 9

FREQUENCY, kc/s


Fig. 10

FREQUENCY, kc/s


Fig. 11
3. Ionospheric Reflection Coefficient for Transverse Magnetic Field -

The ionospheric reflection coefficient ${ }_{\|} \mathrm{R}_{\|}$given by Barber and Crombie [1959] for a transverse terrestrial magnetic field may be written as

$$
\begin{gather*}
\|^{R}=\frac{C-\Delta}{C+\Delta}  \tag{15}\\
\text { where } \Delta=\frac{C^{2}+\left[\frac{1+0}{\delta+\delta^{2}-\gamma^{2}}\right]^{\frac{1}{2}}\left(\delta+\delta^{2}-v^{2}\right)-i \gamma\left[1-C^{2}\right]^{\frac{1}{2}}}{(1+\delta)^{2}-\gamma^{2}} \tag{16a}
\end{gather*}
$$

and $\delta=\mathrm{i} \frac{\omega}{\omega_{\mathrm{r}}}, \gamma=\left\{\begin{array}{l}+\frac{\omega}{\omega_{\mathrm{r}}}\left|\tan \phi_{\mathrm{T}}\right| \text { (East-to-West Propagation) } \\ -\frac{\omega}{\omega_{\mathrm{r}}}\left|\tan \phi_{\mathrm{T}}\right| \text { (West-to-East Propagation) }\end{array}\right.$

Just as in the preceding section, expand $\log _{\|} R_{\|}$in a Taylor series about $C=0$ to get
where

$$
\begin{align*}
& \left\|^{R}\right\|=\exp \left[\sum_{k=0}^{\infty} a_{k} C^{k}\right]  \tag{17}\\
& a_{k}=\frac{1}{k!}\left[\frac { \partial ^ { k } } { \partial C ^ { k } } \left(\log _{\| \|_{\|}} \|_{C=0} .\right.\right. \tag{18}
\end{align*}
$$

The ionospheric reflection coefficient $\|_{\|}{ }_{\|}$can now be approximated by using a few terms of the series (17). The first few Taylor series coefficients for $\log _{\|} \mathrm{R}_{\|}$are

$$
\begin{equation*}
a_{0}=i \pi \tag{19a}
\end{equation*}
$$

$$
\begin{gather*}
a_{1}=-\frac{2\left[(1+\delta)^{2}-\gamma^{2}\right]}{\left[(1+\delta)\left(\delta+\delta^{2}-\gamma^{2}\right)\right]^{\frac{1}{2}}-i \gamma}  \tag{19b}\\
a_{2}=0  \tag{19c}\\
a_{3}=\frac{1}{I 2} a_{1}^{2}\left\{\frac{3}{(1+\delta)^{2}-\gamma^{2}}\left[\frac{\left(\delta+\delta^{2}-\gamma^{2}\right)^{\frac{3}{2}}}{(1+\delta)^{\frac{1}{2}}}+i \gamma\right]+a_{1}\right\} \\
a_{4}=0
\end{gather*}
$$

4. An Approximate Solution of the TM Mode Equation for Perfectly Conducting Ground in the Transverse Case - For a purely transverse and horizontal terrestrial magnetic field, the ionospheric reflection coefficients ${ }_{\|} R_{\perp}$ and ${ }_{\perp} R_{\|}$vanish, so that $T E$ and $T M$ modes are not coupled when the ground is perfectly conducting. The TM mode equation for VLF propagation in a flat earth-ionosphere waveguide is then given by

$$
\begin{equation*}
e^{i 2 k h C}-\|_{\|}=0 \tag{20}
\end{equation*}
$$

If the ionospheric reflection coefficient ${ }_{\|} \mathrm{R}_{\|}$is approximated by using the first two terms of the Taylor series (17) for $\log { }_{\|} R_{\|}$, one has

$$
\|_{\|}^{R_{\|}}=-e^{a_{1} C}
$$

where $a_{1}$ is given by (19b).
With this approximation, the solutions of (20) are

$$
\begin{equation*}
C_{n}=\frac{(2 n-1) \pi}{2 k h+i a_{1}} \quad(n=1,2,3, \ldots) \tag{21}
\end{equation*}
$$

Curves of $1 / \operatorname{Re}\left(S_{n}\right)$ and $-H \operatorname{Im}\left(S_{n}\right)$ vs. $H=\frac{k h}{2 \pi} \quad$ have been computed when $n=1$ for

$$
B=\frac{1}{\mathrm{H}}\left(\frac{\omega^{-}}{\omega_{\mathrm{r}}^{-}}\right)=0.02,0.05,0.10,0.20 \text { and } \phi_{\mathrm{T}}=0^{\circ}, 30^{\circ}, 60^{\circ}
$$

These results are shown in figures 12-19.
Frequency and attenuation scales corresponding to $h=70 \mathrm{~km}$ have been appended.

FREQUENCY, kc/s


Fig. 12

FREQUENCY, kc/s


Fig. 13

FREQUENCY, kc/s


Fig. 14
FREQUENCY, kc/s


Fig. 15

FREQUENCY, kc/s


Fig. 16
FREQUENCY, kc/s


Fig. 17
FREQUENCY, kc/s


Fig. 18

FREQUENCY, kc/s


Fig. 19

## 5. Arbitrarily Dipping Magnetic Field -

In general, the ionospheric reflection coefficients depend on the direction of propagation with respect to the terrestrial magnetic field. Johler [1961] has evaluated ${ }_{\|} R_{\|}{ }_{\perp} R_{\perp}, \|_{\perp} R_{\perp} R_{\|}$for a (real) angle of incidence of $82^{\circ}$, various magnetic dip angles $I$, and directions of propagation $\phi_{a}$ (measured clockwise from north). These results have been used to estimate the effect of variations in I and $\phi_{a}$ on the attenuation and phase of TM waveguide modes. This was done in the following way. First, the ionospheric reflection coefficient ${ }_{\|} R_{\|}$was approximated by an expression of the form

$$
\begin{equation*}
{ }_{\|} R_{\|}=-e^{a C} \tag{22}
\end{equation*}
$$

where the parameter $a$ was chosen so that (22) agrees with the exact value computed by Johler when $C=\cos 82^{\circ}$ [note that when $C=\cos 90^{\circ}$, (22) automatically reduces to the exact value]. Since $\|_{\|} R_{\|} e^{i 2 \pi}$ is equal to Johler's $\mathrm{T}_{\text {ee }}$, this requires that
$\operatorname{Re}(a)=\frac{\log _{e}\left|T_{e e}\right|}{\cos 82^{\circ}}$ and $\operatorname{Im}(a)=\frac{\arg \left(T_{e e^{i-\pi}}\right.}{\cos 82^{\circ}}$.

If coupling between $T M$ and $T E$ modes is neglected (equivalent to setting ${ }_{\|} R_{\perp}={ }_{\perp} R_{\|}=0$ ), the $T M$ mode equation for VLF propagation in a flat earth-ionosphere waveguide becomes

$$
\begin{equation*}
e^{i 2 k h C}-\|_{\|}^{R}=0 \tag{24}
\end{equation*}
$$

Using the expression (22) for ${ }_{\|} R_{\|}$, the solutions of (24) are

$$
C_{n}=\frac{(2 n-1) \pi}{2 k h+i a} \quad(n=1,2,3, \ldots)
$$

Now let

$$
\begin{align*}
P & =\frac{\text { Attenuation with Magnetic Field }}{\text { Attenuation without Magnetic Field }} \\
& =\frac{[\operatorname{Im}(S)] \text { with Magnetic Field }}{[\operatorname{Im}(S)] \text { without Magnetic Field }} \tag{25a}
\end{align*}
$$

and

$$
\begin{align*}
Q & =\frac{\text { Phase Velocity Deviation with Magnetic Field }}{\text { Phase Velocity Deviation without Magnetic Field }} \\
& =\frac{\left[\frac{\mathrm{V}}{\mathrm{c}}-1\right] \text { with Magnetic Field }}{\left[\frac{\mathrm{V}}{\mathrm{C}}-1\right] \text { without Magnetic Field }} \\
& =\frac{\left[\frac{1}{\operatorname{Re}(S)}-1\right] \text { with Magnetic Field }}{\left[\frac{1}{\operatorname{Re}(S)}-1\right] \text { without Magnetic Field }} \tag{25b}
\end{align*}
$$

For the dominant mode ( $n=1$ ), curves of $P$ and $Q$ vs. direction of propagation ( $\phi_{a}$ ) have been computed for:

Magnetic dip angles. $I=0^{\circ}, 45^{\circ}, 84.3^{\circ}$,
Electron densities $, \quad \mathrm{N}=3(10)^{3},(10)^{3}$ electrons $/ \mathrm{cm}^{3}$,
Frequencies, $\quad f=10,22 \mathrm{kc} / \mathrm{s}$.
The results are shown in figures 20 through 24 .

In the case of $I=0^{\circ}, P$ and $Q$ were also computed for $\phi_{a}=90^{\circ}$ and $\phi_{a}=270^{\circ}-$ that is, for a (horizontal) transverse terrestrial magnetic field. In order to make the methods of computation uniform, the exact formula (15) for the ionospheric reflection coefficient $\left\|^{R}\right\|$ was first evaluated for $C=\cos 82^{\circ}$. The determination of $a$ and subsequent calculations were then carried out according to the procedure outlined above.

For $I=90^{\circ}$ (vertical magnetic field), $P$ and $Q$ are independent of the direction of propagation $\phi_{a}$. In this case, $P$ and $Q$ have been plotted as a function of frequency (figure 25).



Fig. 20



Fig. 21



Fig. 22





Fig. 24


Fig. 25
III. Calculations for a Spherical Earth-Ionosphere Waveguide Using the Quasi-longitudinal Approximation

1. The Mode Equation - A simplified form of the mode equation for VLF propagation in a spherical earth-ionosphere waveguide has been given by Wait and Spies [1960]. If the ionospheric reflection coefficient $R_{i}$ is expressed as

$$
\begin{equation*}
R_{i}=-\exp \left[a_{1}\left(C^{2}+\frac{2 h}{a}\right)^{\frac{1}{2}}\right] \tag{26}
\end{equation*}
$$

where the parameter $a_{1}$ is that computed previously (in section II) for the quasi-longitudinal approximation, this mode equation may be written as
$\frac{2}{3} k a\left(C^{2}+\frac{2 h}{a}\right)^{3 / 2}+i a_{1}\left(C^{2}+\frac{2 h}{a}\right)^{1 / 2}+i \log \left[\frac{w_{2}(t)-q w_{2}(t)}{w_{1}^{\prime}(t)-q w_{1}(t)}\right]$

$$
\begin{equation*}
-(4 n-1) \frac{\pi}{2}=0 \tag{27}
\end{equation*}
$$

where $n$ is an integer and $t=-\left(\frac{k a^{2 / 3}}{2}\right)^{2} C^{2}$.

A restriction on this mode equation is that $h / a \ll 1$.
2. A Crude Solution for Perfectly Conducting Ground - When the ground is perfectly conducting ( $q=0$ ), i $\log \left[w_{2}^{\prime}(t) / w_{1}^{\prime}(t)\right]$ may be approximated by

$$
\text { i } \log \left[\frac{w_{2}(0)}{w_{1}(0)}\right]=-\arg \left[w_{2}(0)\right]+\arg \left[w_{1}^{\prime}(0)\right]=\frac{\pi}{3}
$$

and if $\left|C^{2}\right| \ll \frac{2 h}{a}$, if follows that $\left(C^{2}+\frac{2 h}{a}\right)^{3 / 2} \approx\left(\frac{2 h}{a}\right)^{3 / 2}\left[1+\frac{3}{2} \frac{C^{2}}{\frac{2 h}{a}}\right]$ and $\left(C^{2}+\frac{2 h}{a}\right)^{1 / 2} \approx\left(\frac{2 h}{a}\right)^{1 / 2}\left[1+\frac{1}{2} \frac{C^{2}}{\frac{2 h}{a}}\right]$.

The resulting mode equation can be solved at once for $\mathrm{C}^{2}$ to give

$$
\begin{equation*}
C^{2} \approx \frac{(12 \mathrm{n}-5) \frac{\pi}{6}-\frac{2}{3} k a\left(\frac{2 h^{3}}{\mathrm{a}}\right)^{3 / 2}-i a_{1}\left(\frac{2 h^{h}}{\mathrm{a}}\right)^{1 / 2}}{k a\left(\frac{2 h}{\mathrm{a}}\right)^{1 / 2}+i \frac{a_{1}}{2}\left(\frac{2 h}{\mathrm{a}}\right)^{-1 / 2}} . \tag{28}
\end{equation*}
$$

Using a value $h / a=0.01$, curves of $1 / \operatorname{Re}\left(S_{n}\right)$ and $-H \operatorname{Im}\left(S_{n}\right)$ vs. $H=\frac{\mathrm{kh}}{2 \pi}$ have been computed when $n=1$ for

$$
B=\frac{1}{H}\left(\frac{\omega}{\omega_{r}}\right)=0.02,0.05,0.10,0.20 \text { and } \phi_{L}=0^{\circ}, 60^{\circ} .
$$

These are shown in figures 26 through 30. Frequency and attenuation scales corresponding to $\mathrm{h}=70 \mathrm{~km}$ have been appended. Some curves hav also been drawn to show comparisons between solutions for flat and spherical earth-ionosphere waveguides (Fig. 31). The particular curve labelled $[1-h / 2 a)] \times$ Flat Earth is a semi-empirical result which has been used previously for interpreting diurnal change of ionospheric reflection heights [Wait, 1959]. For such purposes, it is a good approximation (here $h / a=0.01$ ).


Fig. 26


Fig. 27


Fig. 28


Fig. 29
FREQUENCY, kc/s


Fig. 30


Fig. 31

An exact solution of the mode equation (27) for perfectly reflecting boundaries (to be described in the following section) indicates that the approximate method described above must be used with considerable caution. For one thing, the assumption that $\left|C^{2}\right| \ll 2 \mathrm{~h} / \mathrm{a}$ is not always justified for $n=1$. In fact, for $n=2$, the reverse is often true, especially for the lower frequencies (around $10 \mathrm{kc} / \mathrm{s}$ )。 Also when the condition $\left|C^{2}\right| \ll 2 \mathrm{~h} / \mathrm{a}$ fails, the approximation i $\log \left[w_{2}^{1}(t) / w_{1}^{\prime}(t)\right] \approx \pi / 3$ is also very poor.

To obtain the corresponding "crude solution" for finitely conducting ground, one may approximate

$$
i \log \left[\frac{w_{2}(t)-q w_{2}(t)}{w_{1}(t)-q w_{1}(t)}\right]
$$

by expanding the logarithm in a Taylor series about $t=0$, then neglecting terms involving $\mathrm{t}^{2}$ and higher powers to get
$i \log \left[\frac{w_{1}(t)-q w_{2}(t)}{w_{1}^{\prime}(t)-q w_{1}(t)}\right] \approx \frac{\pi}{3}+i \log \left[\frac{1-q \frac{w_{2}(0)}{w_{1}(0)}}{1-q \frac{w_{1}(0)}{w_{1}^{1}(0)}}\right]$

$$
\begin{equation*}
-i\left(\frac{\mathrm{ka}}{2}\right) \frac{\mathrm{q}^{2}\left(\frac{\mathrm{w}_{1}(0)}{\mathrm{w}_{1}^{\prime}(0)}-\frac{\mathrm{w}_{2}(0)}{\mathrm{w}_{2}^{1}(0)}\right) C^{2}}{\left(1-\mathrm{q} \frac{\mathrm{w}_{1}(0)}{\mathrm{w}_{1}^{1}(0)}\right)\left(1-\mathrm{q} \frac{\mathrm{w}_{2}(0)}{\mathrm{w}_{2}^{1}(0)}\right)} . \tag{29}
\end{equation*}
$$

Using this approximation, the method of solution described above gives

$$
\begin{equation*}
C^{2} \approx \frac{(12 n-5) \frac{\pi}{6}-\frac{2}{3} k a\left(\frac{2 h^{3 / 2}}{a}-i a_{1}\left(\frac{2 h^{1 / 2}}{a}\right)^{-1 / 2}-i \delta_{o}\right.}{k a\left(\frac{2 h}{a}\right)+i \frac{u_{1}}{2}\left(\frac{2 h^{-1}}{a}\right)^{2 / 3}-i\left(\frac{k a}{2}\right)^{2} \delta_{1}} \tag{30}
\end{equation*}
$$

where $\delta_{0}=\log \left[\frac{1-q \frac{w_{2}(0)}{w_{2}^{1}(0)}}{1-q \frac{w_{1}(0)}{w_{1}(0)}}\right]$ and $\delta_{1}=\frac{q^{2}\left(\frac{w_{1}(0)}{w_{1}(0)}-\frac{w_{2}(0)}{w_{2}^{1}(0)}\right)}{\left(1-q \frac{w_{1}(0)}{w_{1}^{1}(0)}\right)\left(1-q \frac{w_{2}(0)}{w_{2}^{1}(0)}\right)}$.
However, further effort along this line does not seem warranted, since there still exists the difficulty that $\left|C^{2}\right| \ll 2 h / a$ is not always true (nor is the approximation (29) always realistic).
3. Solution for Perfectly Reflecting Boundaries - When $R_{g}=+1$ ( $\sigma=\infty$ or $q=0$ ) and $R_{i}=-1\left(a_{1}=0\right)$, the mode equation (27) may be written as

$$
\begin{equation*}
\left(C^{2}+\frac{2 h^{3 / 2}}{a}\right)^{\frac{1}{3} k a}\left[(4 n-1) \frac{\pi}{2}+2 \tan ^{-1}\left(\frac{v^{\prime}(t)}{u^{\prime}(t)}\right)\right] \tag{32}
\end{equation*}
$$

For perfectly reflecting boundaries, the modes propagate without attenuation, so that real solutions of (32) are desired. For real $t$,

$$
\begin{equation*}
i \log \left[\frac{w_{2}(t)}{w_{1}^{\prime}(t)}\right]=-2 \tan ^{-1}\left(\frac{v^{\prime}(t)}{u^{\prime}(t)}\right) \tag{33}
\end{equation*}
$$

where the inverse tangent is a continuous function ${ }^{1}$ of $t$ such that

$$
\tan ^{-1}\left(\frac{v^{\prime}(0)}{u^{\prime}(0)}\right)=-\frac{\pi}{6}
$$

First approximations to the solutions of (32) were obtained for $\mathrm{n}=1$ and $\mathrm{n}=2$ by graphical means -- plotting the right- and left-hand members of $(32)$ vs. $C^{2}$, then reading off the abscissae of the points of intersection. Two further approximations were then obtained by using the "method of false position."

In appendix $A$, this function is tabulated to six decimal places at
intervals of 0.1 from $t=-10.0$ to $t=+5.0$. intervals of 0.1 from $t=-10.0$ to $t=+5.0$ 。

Curves of $\left(\frac{v}{c}-1\right)$ vs. frequency have been computed for $h=60,70$, $80,90,100 \mathrm{~km}$. The frequency range covered extends from $8 \mathrm{kc} / \mathrm{s}$ to $30 \mathrm{kc} / \mathrm{s}$. [Using graphical means, curves have been interpolated for $h=65,75,85,95 \mathrm{~km}$ 。] These results are shown in figures 32 and 33 .

For $h=60,80,100 \mathrm{~km}$, curves of $\left(\frac{v}{c}-1\right) \mathrm{vs}$. frequency were also computed using the crude method of section 2 [set $\alpha_{1}=0$ in (28)] when $n=1$. A comparison of the crude solution with the exact solution of this section is shown in figure 34.
4. Newton's Method for Solving the Mode Equation - Let $z=C^{2}$ and write the mode equation (27) as

$$
\begin{equation*}
F(z)=0 \tag{34}
\end{equation*}
$$

where $\quad F(z)=\frac{2}{3} k a\left(z+\frac{2 h^{3 / 2}}{a}\right)^{\left.\text {i } a_{1}\left(z+\frac{2 h^{l / 2}}{a}\right)^{2}\right) ~}$

$$
\begin{equation*}
+i \log \left|\frac{w_{2}(t)-q w_{2}(t)}{w_{1}^{\prime}(t)-q w_{1}(t)}\right|-(4 n-1) \frac{\pi}{2} \tag{35}
\end{equation*}
$$

and

$$
t=-\left(\frac{k a}{2}\right)^{2 / 3} z
$$

According to Newton's method, if $z_{o}$ is an approximate root of (34), a next approximation $z_{1}$ is given by

$$
z_{1}=z_{0}+\Delta z
$$

where

$$
\Delta z=-\frac{F\left(z_{0}\right)}{F^{\prime}\left(z_{0}\right)}
$$



Fig. 32


Fig. 33


Fig. 34

Differentiating (35) with respect to $z$ and using the relation

$$
w_{1}^{\prime}(t) w_{2}(t)-w_{1}(t) w_{2}(t)=2 i
$$

gives $\quad F^{\prime}(z)=k a\left(z+\frac{2 h}{a}\right)^{1 / 2}+i \frac{a_{1}}{2}\left(z+\frac{2 h}{a}\right)^{-1 / 2}$

$$
+\left(\frac{k a}{2}\right)^{2 / 3} \frac{2\left(t-q^{2}\right)}{\left[w_{1}(t)-q w_{1}(t)\right]\left[w_{2}(t)-q w_{2}(t)\right]}
$$

so that $\Delta z=$

$$
-\frac{\frac{2}{3} k a\left(z_{o}+\frac{2 h}{a}\right)^{3 / 2}+i a_{1}\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}+i \log \left[\frac{w_{2}^{\prime}\left(t_{o}\right)-q w_{2}\left(t_{o}\right)}{w_{1}\left(t_{o}\right)-q w_{1}\left(t_{o}\right)}\right]-(4 n-1) \frac{\pi}{2}}{k a\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}+i \frac{a_{1}}{2}\left(z_{o}+\frac{2 h^{-\frac{1}{2}}}{a}\right)^{2\left(t_{o}-q^{2}\right)}+\left(\frac{k a}{2}\right)^{2 / 3} \frac{\left.2 w_{1}^{\prime}\left(t_{o}\right)-q w_{1}\left(t_{o}\right)\right]\left[w_{2}^{\prime}\left(t_{o}\right)-q w_{2}\left(t_{o}\right)\right]}{}}
$$

where

$$
\begin{equation*}
t_{o}=-\left(\frac{k a^{2 / 3}}{2}\right)^{z_{o}} \tag{36}
\end{equation*}
$$

Further approximations can be obtained by successive applications of Newton's method.
5. Solution for Perfectly Reflecting Ionosphere and Finitely Conducting Ground - For a perfectly reflecting ionosphere, $a_{1}=0$, so that the mode equation (27) becomes
where $z=C^{2}$. If $z_{o}$ is an approximate root of (37), Newton's method then gives a next approximation, $z_{1}=z_{o}+\Delta z$
where

$$
\Delta z=-\frac{\frac{2}{3} k a\left(z_{0}+\frac{2 h}{a}\right)^{3 / 2}+i \log \frac{r^{w_{2}^{\prime}\left(t_{0}\right)-q w_{2}\left(t_{0}\right)}}{k a\left(z_{0}+\frac{2 h}{a}\right)^{\frac{1}{2}}+\left(\frac{k a}{2}\right)}{ }^{2 / 3}\left(t_{0}\right)-q w_{1}\left(t_{0}\right)}{} \frac{2\left(t_{0}^{-q}\right)}{\left[w_{1}^{\prime}\left(t_{0}\right)-q w_{1}\left(t_{0}\right)\right]\left[w_{2}^{\prime}\left(t_{0}\right)-q w_{2}\left(t_{0}\right)\right]}
$$

If, for the first approximation, one chooses $z_{o}$ to be the solution of the mode equation (32) for perfectly reflecting boundaries, so that

$$
\frac{2}{3} \mathrm{ka}\left(z_{o}+\frac{2 h}{a}\right)^{3 / 2}+i \log \left[\frac{w_{2}^{\prime}\left(t_{0}\right)}{w_{1}^{\prime}\left(t_{0}\right)}\right]-(4 n-1) \frac{\pi}{2}=0
$$

then in the expression for the second approximation $z_{1}=z_{o}+\Delta z$, $\Delta \mathrm{z}$ is given by

$$
\begin{equation*}
\Delta z=\frac{i \log \left[\frac{w_{2}^{\prime}\left(t_{0}\right)}{w_{1}\left(t_{0}\right)}\right]-i \log \left[\frac{w_{2}^{1}\left(t_{0}\right)-q w_{2}\left(t_{0}\right)}{w_{1}^{\prime}\left(t_{0}\right)-q w_{1}\left(t_{0}\right)}\right]}{2\left(t_{0}-q^{2}\right)} \operatorname{ka}^{\frac{1}{2}}+\left(\frac{k a}{2}\right)^{2 / 3} \frac{2\left(w_{1}^{\prime}\left(t_{0}\right)-q w_{1}\left(t_{0}\right)\right]\left[w_{2}^{\prime}\left(t_{0}\right)-q w_{2}\left(t_{0}\right)\right]}{} . \tag{39}
\end{equation*}
$$

For any further approximations, however, one must use (38) to evaluate $\Delta z$.

Choosing $z_{o}$ to be the solution of the mode equation for perfectly reflecting boundaries, a single application of Newton's method has been used to compute curves of $\left(\frac{\mathrm{v}}{\mathrm{c}}-1\right)$ vs. frequency and attenuation $(\mathrm{db} / 1000 \mathrm{~km})$ vs. frequency for $\mathrm{n}=1$ when $\sigma_{\mathrm{g}}=5 \times 10^{-3} \mathrm{mho} / \mathrm{m}$ and $h=60,70,80,90,100 \mathrm{~km}$. The frequency range covered extends from $8 \mathrm{kc} / \mathrm{s}$ to $30 \mathrm{kc} / \mathrm{s}$. Using graphical means, curves have been interpolated for $h=65,75,85,95 \mathrm{~km}$. These results are shown in figures 35 and 36 .
6. Solution for Imperfectly Reflecting Ionosphere and Perfectly Conducting Ground - For a perfectly conducting ground $q=0\left(\sigma_{g}=\infty\right)$, so that the mode equation (27) becomes

$$
\begin{equation*}
\frac{2}{3} k a\left(z+\frac{2 h}{a}\right)^{3 / 2}+i a_{1}\left(z+\frac{2 h}{a}\right)^{1 / 2}+i \log \left[\frac{w_{2}^{\prime}(t)}{w_{1}^{\prime}(t)}\right]-(4 n-1) \frac{\pi}{2}=0 \tag{40}
\end{equation*}
$$

where $z=C^{2}$. If $z_{o}$ is an approximate root of (40), Newton's method then gives a next approximation $z_{1}=z_{0}+\Delta z$
where $\Delta z=-\frac{\frac{2}{3} k a\left(z_{o}+\frac{2 h^{3 / 2}}{a}\right)^{2}+i a_{1}\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}+i \log \left[\frac{w_{2}^{\prime}\left(t_{0}\right)}{w_{1}^{\prime}\left(t_{0}\right.}\right]-(4 n-1) \frac{\pi}{2}}{k a\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}+i \frac{a_{1}}{2}\left(z_{o}+\frac{2 h^{-\frac{1}{2}}}{a}\right)^{2}+\left(\frac{k a^{2} / 3}{2}\right) \frac{2 t_{o}}{w_{1}^{\prime}\left(t_{0}\right) w_{2}^{1}\left(t_{0}\right)}}$.

If, for the first approximation, one chooses $z_{o}$ to be the solution of the mode equation (32) for perfectly reflecting boundaries, so that

$$
\frac{2}{3} k a\left(z_{o}+\frac{2 h^{3 / 2}}{a}\right)^{2}+i \log \left[\frac{w_{2}^{\prime}\left(t_{0}\right)}{w_{1}^{\prime}\left(t_{0}\right)}\right]-(4 n-1) \frac{\pi}{2}=0
$$




Fig. 36
then in the expression for the second approximation $z_{1}=z_{o}+\Delta z$, $\Delta z$ is given by

$$
\begin{equation*}
\Delta z=-\frac{i a_{i}\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}}{k a\left(z_{o}+\frac{2 h}{a}\right)^{\frac{1}{2}}+i \frac{a_{1}}{2}\left(z_{o}+\frac{2 h}{a}\right)^{-\frac{1}{2}}+\left(\frac{k a}{2}\right)^{2 / 3} \frac{2 t_{0}}{\left[u^{\prime}\left(t_{0}\right)\right]^{2}+\left[v^{\prime}\left(t_{0}\right)\right]^{2}}} . \tag{42}
\end{equation*}
$$

For any further approximations, however, one must use (41) to evaluate $\Delta z$.

Choosing $z_{o}$ to be the solution of the mode equation for perfectly reflecting boundaries, a single application of Newton's method has been used to compute curves of $\left(\frac{V}{c}-1\right) \mathrm{vs}$. frequency and attenuation $(\mathrm{db} / 1000 \mathrm{~km})$ vs. frequency for $\mathrm{n}=1$ and $\mathrm{n}=2$ in the absence of a terrestrial magnetic field when $\omega_{r}=2 \times 10^{5}$ and $h=60,70,80,90$, 100 km . Again, the frequency range covered extends from $8 \mathrm{kc} / \mathrm{s}$ to $30 \mathrm{kc} / \mathrm{s}$. Using graphical means, $\left(\frac{\mathrm{V}}{\mathrm{c}}-\mathrm{l}\right) \mathrm{vs}$. frequency curves have been interpolated for $\mathrm{h}=65,75,85,95 \mathrm{~km}$ for $\mathrm{n}=2$. Attenuation vs. frequency curves have been interpolated for these same values of h. These results are shown in figures 37 through 40 .
7. Solution for Imperfectly Reflecting Ionosphere and Finitely Conducting Ground - Using Newton ${ }^{8}$ s method, solutions of mode equation (27) have been obtained for $\mathrm{n}=1$ when $\mathrm{h}=70 \mathrm{~km}, \omega_{\mathrm{r}}=2 \times 10^{5}$, and $\sigma_{g}=1,2,5,10,20$ millimhos $/$ meter. Again, the frequency range extended from 8 to $30 \mathrm{kc} / \mathrm{s}$. Initially, $\mathrm{z}_{\mathrm{o}}$ was chosen to be the solution of (27) when $\omega_{r}=2 \times 10^{5}$ and $\sigma_{g}=\infty$. However, for the smaller conductivities at the lower frequencies, it was found that such a choice of starting values did not result in convergence, so the computing technique was modified in the following manner. The solutions for $\sigma_{\mathrm{g}}=20$ millimhos/meter were obtained first by choosing $\mathrm{z}_{\mathrm{o}}$ to be the solution of mode equation (27) for perfectly conducting ground and
$\omega_{r}=2 \times 10^{5}$. The solutions for $\sigma_{g}=20$ were then used as starting values to obtain solutions for $\sigma_{g}=10$ and these solutions were in turn used as starting values for $\sigma_{g}=5$, and so on. In every case, one application of Newton ${ }^{\prime}$ s method was sufficient to produce adequate convergence, though a second application was carried out as a check on the computations. The results of these calculations, in the form of curves of $\left(\frac{v}{c}-1\right)$ vs. frequency and attenuation ( $\mathrm{db} / 1000 \mathrm{~km}$ ) vs. frequency, are shown in figures 41 and 42. Calculations for $\sigma_{g}=4 \mathrm{mhos} / m e t e r$ (corresponding to sea water) were also carried out, but the results were practically identical to those for $\sigma_{g}=\infty$.


Fig. 37


Fig. 38
ATTENUATION, db/IOOOkm


Fig. 39


Fig. 40


Fig. 41
10


ATTENUATION, db/IOOO km
8

7
os
or

3

2
 0
IV. Further Calculations for a Spherical Earth-Ionosphere Waveguide 1. A More Accurate Form of the Mode Equation - A more accurate form of the mode equation for VLF propagation in a spherical earthionosphere waveguide is given by [Wait, 1960]

$$
\begin{equation*}
\left[\frac{w_{2}^{\prime}\left(t_{a}\right)-q w_{2}\left(t_{a}\right)}{w_{1}^{\prime}\left(t_{a}\right)-q w_{1}\left(t_{a}\right)}\right]\left[\frac{w_{1}^{\prime}\left(t_{c}\right)+q w_{1}\left(t_{c}\right)}{w_{2}^{\prime}\left(t_{c}\right)+q w_{2}\left(t_{c}\right)}\right] e^{-i \Phi}=e^{-i 2 n \pi} \tag{43}
\end{equation*}
$$

where

$$
\begin{aligned}
\Phi & =2\left(\gamma_{c}-\gamma_{a}\right)-2\left(\rho_{c}-\rho_{a}\right), & & (\operatorname{Re}(S)<1) \\
& =2 \gamma_{c}-2 \rho_{c} \quad, & & \left(1<\operatorname{Re}(S)<1+\frac{h}{a}\right)
\end{aligned}
$$

$\rho_{a}=\frac{k a S}{3}\left[\frac{(k a)^{2}}{(k a S)^{2}}-1\right]^{3 / 2} \quad \gamma_{a}=\int_{k a S}^{k a}\left[1-\frac{(k a S)^{2}}{x^{2}}\right]^{1 / 2} d x$
$\rho_{C}=\frac{\mathrm{ka} S}{3}\left[\frac{(\mathrm{kc})^{2}}{(\mathrm{ka} \mathrm{S})^{2}}-1\right]^{73 / 2}$

$$
\gamma_{c}=\int_{k a S}^{k c}\left[1-\frac{(k a S)^{2}}{x^{2}}\right]^{1 / 2} d x
$$

and

$$
\begin{array}{ll}
\rho_{a}=\frac{2}{3}\left(-t_{a}\right)^{3 / 2} & t_{a}=-\left(\frac{3}{2} \rho_{a}\right)^{2 / 3}, \\
\rho_{c}=\frac{2}{3}\left(-t_{c}\right)^{3 / 2} & t_{c}=-\left(\frac{3}{2} \rho_{c}\right)^{2 / 3} \\
c=h+a . &
\end{array}
$$

Equation (43) is not restricted by the condition $h / a \ll 1$. By carrying out the above integrations and a certain amount of algebra, this mode equation may be written (if $z=C^{2}$ )

$$
\begin{equation*}
i \log \left[\frac{w_{2}^{\prime}\left(t_{a}\right)-q w_{2}\left(t_{a}\right)}{w_{1}^{\prime}\left(t_{a}\right)-q w_{1}\left(t_{a}\right)}\right]+i \log \left[\frac{w_{1}^{\prime}\left(t_{c}\right)+q_{i} w_{1}\left(t_{i c}\right)}{w_{2}^{\prime}\left(t_{c}\right)+q_{i} w_{2}\left(t_{c}\right)}\right]+\Phi-2 n \pi=0 \tag{44}
\end{equation*}
$$

where $\quad t_{a}=-\left(\frac{\mathrm{ka}}{2}\right)^{2 / 3} \frac{z}{(1-z)^{2 / 3}} \quad, \quad t_{c}=-\left(\frac{k a}{2}\right)^{2 / 3} \frac{z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}}{(1-z)^{2 / 3}}$,
and
$\left.\Phi=2 \mathrm{ka}\left\{\sqrt{z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}-\sqrt{z}+\sqrt{1-z}\left[\cos ^{-1}(\sqrt{1-z})\right.}-\cos ^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right)\right]\right\}$

$$
\begin{gathered}
-\frac{2 k a}{3} \frac{1}{1-z}\left[\left(z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}\right)^{3 / 2}-z^{3 / 2}\right] \cdot(\operatorname{Re}(z)>0) \\
=2 \mathrm{ka}\left[\sqrt{z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}}-\sqrt{1-z} \cos ^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right)\right]-\frac{2 k a}{3} \frac{\left(z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}\right)^{3 / 2}}{1-z}
\end{gathered}
$$

$$
(\operatorname{Re}(z)<0) .
$$

The (complex) inverse cosines above are made unique by requiring them to vanish as their arguments approach unity. Since the arguments are in fact close to one, an infinite series representation convenient for this range is obtained by considering $\cos ^{-1}(1-\mathrm{u})$ where $|u|$ is small. Now
$\cos ^{-1}(1-u)=\frac{\pi}{2}-\sin ^{-1}(1-u)=\frac{\pi}{2}-\int_{0}^{1-u} \frac{d t}{\sqrt{1-t^{2}}}$,
so

$$
\cos ^{-1}(1-u)=\frac{\pi}{2}-\left[\int_{0}^{1} \frac{d t}{\sqrt{1-t^{2}}}-\int_{1-u}^{1} \frac{d t}{\sqrt{1-t^{2}}}\right]=\int_{1-u}^{1} \frac{d t}{\sqrt{1-t^{2}}}
$$

Introducing a new variable of integration by means of the relation $\mathrm{w}=1-\mathrm{t}$, one gets

$$
\cos ^{-1}(1-u)=-\int_{u}^{0} \frac{d w}{\sqrt{2 w-w^{2}}}=\frac{1}{\sqrt{2}} \int_{0}^{u} w^{-\frac{1}{2}}\left(1-\frac{w}{2}\right)^{-\frac{1}{2}} d w .
$$

Expanding the right-hand integrand into an infinite series (using the binomial theorem) and integrating term-by-term gives the desired result:

$$
\begin{equation*}
\cos ^{-1}(1-u)=\sqrt{2 u}\left[1+\frac{1}{1!(3)(2)^{2}} u+\frac{(1)(3)}{2!(5)(2)^{4}} u^{2}+\frac{(1)(3)(5)}{3!(7)(2)^{6}} u^{3}+\ldots\right] . \tag{45}
\end{equation*}
$$

This result is now used to obtain infinite series for

$$
\cos ^{-1}(\sqrt{1-z}) \quad \text { and } \quad \cos ^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right)
$$

Expanding $\sqrt{1-\mathrm{z}}$ into an infinite series (again using the binomial theorem), one has

$$
\begin{equation*}
\cos ^{-1}(\sqrt{1-z})=\cos ^{-1}\left(1-u_{1}\right) \tag{46a}
\end{equation*}
$$

where $\quad u_{1}=\frac{1}{1!(2)} z+\frac{1}{2!(2)^{2}} z^{2}+\frac{(1)(3)}{3!(2)^{3}} z^{3}+\frac{(1)(3)(5)}{4!(2)^{4}} z^{4}+\ldots(46 \mathrm{~b})$

Now $\frac{\sqrt{1-z}}{1+\frac{h}{a}}=\left(1-\frac{\frac{h}{a}}{1+\frac{h}{a}} ;\left(1-u_{1}\right)=1-\frac{1}{1+\frac{h}{a}}\left[\frac{h}{a}-\frac{h}{a} u_{1}+u_{1}\left(1+\frac{h}{a}\right)\right]\right.$.

Thus

$$
\begin{equation*}
\cos ^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right)=\cos ^{-1}\left(1-u_{2}\right) \tag{47a}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{2}=\left(1+\frac{h}{a}\right)^{-1}\left(\frac{h}{a}+u_{1}\right) . \tag{47b}
\end{equation*}
$$

2. Solution for Perfectly Reflecting Boundaries - When the ground is perfectly reflecting, $q=0$; when the ionosphere is perfectly reflecting, $q_{i}=\infty$. The mode equation (44) then becomes

$$
\begin{equation*}
i \log \left[\frac{w_{2}^{\prime}\left(t_{a}\right)}{w_{1}\left(t_{a}\right)}\right]+i \log \left[\frac{w_{1}\left(t_{c}\right)}{w_{2}\left(t_{c}\right)}\right]+\Phi-2 n \pi=0 . \tag{48}
\end{equation*}
$$

Again, real solutions of (48) are desired since the modes propagate without attenuation when the boundaries are perfectly reflecting. For real arguments,

$$
\begin{equation*}
i \log \left[\frac{w_{2}^{\prime}\left(t_{a}\right)}{w_{1}\left(t_{a}\right)}\right]=-2 \tan ^{-1}\left[\frac{v^{\prime}\left(t_{a}\right)}{u^{\prime}\left(t_{a}\right)}\right] \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.i \log \left\lvert\, \frac{w_{1}\left(t_{c}\right)}{w_{2}\left(t_{c}\right)}\right.\right]=2 \tan ^{-1}\left[\frac{v\left(t_{c}\right)}{u\left(t_{c}\right)}\right] \tag{50}
\end{equation*}
$$

where the inverse tangents are continuous functions of $t$ such that

$$
\tan ^{-1}\left[\frac{v^{\prime}(0)}{u^{\prime}(0)}\right]=-\frac{\pi}{6} \quad \text { and } \quad \tan ^{-1}\left[\frac{v(0)}{u(0)}\right]=+\frac{\pi}{6}
$$

The mode equation for perfectly reflecting boundaries can now be written as

$$
\begin{equation*}
\tan ^{-1}\left[\frac{v\left(t_{c}\right)}{u\left(t_{c}\right)}\right]-\tan ^{-1}\left[\frac{v^{\prime}\left(t_{a}\right)}{u^{\prime}\left(t_{a}\right)}\right]+\frac{1}{2} \Phi-n \pi=0 . \tag{51}
\end{equation*}
$$

where $t_{a}, t_{c}$ and $\Phi$ are defined in the previous section [following (44)].

[^0]The solutions of the mode equation (32) should be a fair approximation to the solutions of the mode equation (48) above. With these solutions as a guide, a single application of the "method of false position" has been used to obtain a further approximation for $n=1$ when $h=60,100 \mathrm{~km}$. These results are shown in figure 43.
As expected, they differ but little from those obtained by solving the mode equation (32).
3. Newton's Method for Solving the Mode Equation - Write the mode equation (44) as

$$
\begin{equation*}
F(z)=0 \tag{52}
\end{equation*}
$$

where $F(z)=i \log \left[\frac{w_{2}^{\prime}\left(t_{a}\right)-q w_{2}\left(t_{a}\right)}{w_{1}\left(t_{a}\right)-q w_{1}\left(t_{a}\right)}\right]+i \log \left[\frac{w_{1}^{\prime}\left(t_{c}\right)+q_{i} w_{1}\left(t_{c}\right)}{w_{2}^{\prime}\left(t_{c}\right)+q_{i} w_{2}\left(t_{c}\right)}\right]+\Phi(z)-2 n \pi$
and $t_{a}, t_{c}, \Phi(z)$ are defined following (44). Then according to Newton's method, if $z_{o}$ is an approximate root of (53), a next approximation $z_{1}$ is given by

$$
z_{1}=z_{0}+\Delta z
$$

$$
\Delta z=-\frac{F\left(z_{o}\right)}{F^{\prime}\left(z_{o}\right)}
$$

Differentiating (53) with respect to $z$ and using the relation

$$
\mathrm{w}_{1}^{\prime}(\mathrm{t}) \mathrm{w}_{2}(\mathrm{t})-\mathrm{w}_{1}(\mathrm{t}) \mathrm{w}_{2}^{\prime}(\mathrm{t})=2 \mathrm{i}
$$

gives

$$
\begin{aligned}
F^{\prime}(z)= & \frac{2\left(\frac{k a}{2}\right)^{2 / 3}}{(1-z)(1-z)^{2 / 3}}\left\{\frac{\left(t_{a}-q^{2}\right)\left(1-\frac{1}{3} z\right)}{\left[w_{1}^{\prime}\left(t_{a}\right)-q w_{1}\left(t_{a}\right)\right]\left[w_{2}^{\prime}\left(t_{a}\right)-q w_{2}\left(t_{a}\right)\right]}\right. \\
& \left.-\frac{\left(t_{c}-q_{i}^{2}\right)(1-z)+\frac{2}{3}\left(z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}\right)}{\left[w_{1}^{\prime}\left(t_{c}\right)+q_{i} w_{1}\left(t_{c}\right)\right]\left[w_{2}^{\prime}\left(t_{c}\right)+q_{i} w_{2}\left(t_{c}\right)\right]}\right\}+\frac{d \Phi}{d z}
\end{aligned}
$$

where

$$
\begin{aligned}
& \frac{d \Phi}{d z}=k a\left\{\frac{1}{\sqrt{1-z}}\left[\cos ^{-1}\left(\frac{\sqrt{1-} \bar{z}}{1+\frac{h}{a}}\right)-\cos ^{-1}(\sqrt{1-z})\right]\right. \\
& \left.-\frac{1}{\Gamma-z}\left[\sqrt{z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}}-\sqrt{z}\right]\right\}+\frac{2}{3} k a \frac{1}{(1-z)^{2}}\left[\left(z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}\right)^{3 / 2}-z^{3 / 2}\right]
\end{aligned}
$$

$$
(\operatorname{Re}(z)>0)
$$

$$
\begin{aligned}
& =k a\left\{\frac{1}{\sqrt{1-z}} \cos ^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right)-\frac{1}{1-z} \sqrt{2+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}}\right\} \\
& \\
& \quad+\frac{2}{3} k a \frac{1}{(1-z)^{2}}\left(z+\frac{2 h}{a}+\frac{h^{2}}{a^{2}}\right) . \quad(\operatorname{Re}(z)<0)
\end{aligned}
$$



Fig. 43

Appendix A
Table of Inverse Tangents of Airy Functions ${ }^{\dagger}$

| t | $\tan ^{-1}\left[\frac{v(t)}{u(t)}\right]$ | $\delta_{m}^{2}$ | $\tan ^{-1}\left\lceil\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]$ | $\delta_{m}^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0.523599 | +4 603 | -0.523 599 | + 11929 |
| 0.1 | 0.462765 | 4595 | -0.517681 | + 11671 |
| 0.2 | 0.406521 | 4560 | -0.500 215 | + 10745 |
| 0.3 | 0.354832 | 4500 | -0.472 147 | + 9044 |
| 0.4 | 0.307637 | 4409 | -0.435 170 | + 5612 |
| 0.5 | 0.264845 | 4288 | -0.391666 | + 3720 |
| 0.6 | 0.226335 | 4134 | -0.344 452 | + 771 |
| 0.7 | 0.191953 | 3950 | -0.296 397 | - 1797 |
| 0.8 | 0.161515 | 3734 | -0.250 015 | - 3690 |
| 0.9 | 0.134806 | 3492 | -0.207 183 | - 4825 |
| 1.0 | 0.111585 | 3228 | -0.169 050 | - 5276 |
| 1.1 | 0.091589 | 2946 | -0.136 099 | - 5215 |
| 1.2 | 0.074538 | 2659 | -0.108 305 | - 4839 |
| 1. 3 | 0.060145 | 2364 | -0.085 318 | - 4289 |
| 1.4 | 0.048117 | 2077 | -0.066 609 | - 3679 |
| 1.5 | 0.038168 | 1799 | -0.051582 | - 3086 |
| 1.6 | 0.030021 | 1541 | -0.039 649 | - 2535 |
| 1.7 | 0.023418 | 1298 | -0.030 264 | - 2056 |
| 1.8 | 0.018118 | 1082 | -0.022948 | - 1646 |
| 1.9 | 0.013905 | 893 | -0.017290 | - 1302 |
| 2.0 | 0.010589 | 724 | -0.012946 | - 1023 |
| 2.1 | 0.008002 | 583 | -0.009 635 | 796 |
| 2.2 | 0.006002 | 463 | -0.007 128 | 614 |
| 2.3 | 0.004469 | 365 | -0.005 242 | 472 |
| 2.4 | 0.003304 | 284 | -0.003 833 | 357 |
| 2.5 | 0.002426 | 220 | -0.002 786 | 272 |

$\dagger$
These functions, expressed in degrees, have also been extensively tabulated by Miller [1946]. Note that

$$
\tan ^{-1}\left[\frac{v(t)}{u(t)}\right]=x(t) \quad \text { and } \quad \tan ^{-1}\left[\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]=\psi(t)
$$

where $\chi(\mathrm{t})$ and $\psi(\mathrm{t})$ are the functions given in Miller's table. (The authors are grateful to Nelson A. Logan for pointing out this relationship.) (Also, see footnote on page 76.)

| t | $\tan ^{-1}\left[\frac{\mathrm{v}(\mathrm{t})}{\mathrm{u}(\mathrm{t})}\right]$ | $\delta^{2}$ | $\tan ^{-1}\left[\frac{v^{8}(t)}{u^{8}(t)}\right]$ | $\delta^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 2.5 | 0.002426 | $+222$ | - 0.002786 | - 275 |
| 2.6 | 0.001770 | 168 | -0.002014 | - 206 |
| 2.7 | 0.001282 | 129 | - 0.001448 | - 153 |
| 2.8 | 0.000923 | 096 | -0.001 035 | - 113 |
| 2.9 | 0.000660 | 073 | -0.000735 | - 085 |
| 3.0 | 0.000470 | 052 | -0.000 520 | - 060 |
| 3.1 | 0.000332 | 039 | -0.000 365 | - 045 |
| 3.2 | 0.000233 | 029 | -0.000255 | - 033 |
| 3.3 | 0.000163 | 020 | - 0.000178 | - 022 |
| 3.4 | 0.000113 | 015 | -0.000 123 | - 017 |
| 3.5 | 0.000078 | 011 | -0.000 085 | - 011 |
| 3.6 | 0.000054 | 007 | - 0.000058 | - 008 |
| 3.7 | 0.000037 | 005 | - 0.000039 | - 007 |
| 3.8 | 0.000025 | 004 | - 0.000027 | - 003 |
| 3.9 | 0.000017 | 002 | -0.000 018 | - 003 |
| 4.0 | 0.000011 | 003 | - 0.000012 | - 002 |
| 4.1 | 0.000008 | 000 | - 0.000008 | - 001 |
| 4.2 | 0.000005 | 001 | -0.000 005 | - 002 |
| 4.3 | 0.000003 | 001 | - 0.000004 | + 001 |
| 4.4 | 0.000002 | 000 | - 0.000002 | - 002 |
| 4.5 | 0.000001 | 001 | - 0.000002 | + 001 |
| 4.6 | 0.000001 | 000 | - 0.000001 | - 001 |
| 4.7 | 0.000001 | - 001 | - 0.000001 | 000 |
| 4.8 | 0.000000 | + 001 | 0.000000 | - 001 |
| 4.9 | 0.000000 | 000 | 0.000000 | 000 |
| 5.0 | 0.000000 | 000 | 0.000000 | 000 |


| t | $\tan ^{-1}\left[\frac{\mathrm{v}(\mathrm{t})}{\mathrm{u}(\mathrm{t})}\right]$ | $\delta_{m}^{2}$ | $\tan ^{-1}\left[\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]$ | $\delta_{m}^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0.523599 | + 4603 | -0.523 599 | + 11929 |
| -0.1 | 0.589033 | 4595 | -0.517678 | 11698 |
| -0.2 | 0.659059 | 4568 | -0.500 116 | 11159 |
| -0.3 | 0.733650 | 4525 | -0.471424 | 10463 |
| -0.4 | 0.812764 | 4474 | -0.432 278 | 9716 |
| -0.5 | 0.896350 | 4410 | -0.383414 | 8977 |
| -0.6 | 0.984345 | 4343 | -0.325 564 | 8286 |
| -0.7 | 1.076682 | 4270 | -0.259417 | 7655 |
| -0.8 | 1. 173288 | 4192 | -0.185602 | 7096 |
| -0.9 | 1.274086 | 4115 | -0.104 680 | 6594 |
| -1.0 | 1.378999 | 4036 | -0.017151 | 6163 |
| -1.. ${ }^{1}$ | 1.487948 | 3958 | +0.076550 | 5778 |
| -1.2 | 1.660855 | 3879 | +0.176 038 | 5442 |
| -1. 3 | 1.717641 | 3801 | +0.280976 | 5147 |
| -1. 4 | 1.838229 | 3728 | +0.391068 | 4888 |
| -1. 5 | 1.962545 | 3655 | 0.506054 | 4659 |
| -1.6 | 2.090516 | 3582 | 0.625704 | 4455 |
| -1.7 | 2.222070 | 3515 | 0.749813 | 4274 |
| -1.8 | 2.357139 | 3447 | 0.878199 | 4110 |
| -1.9 | 2.495656 | 3384 | 1.010698 | 3964 |
| -2.0 | 2.637557 | 3320 | 1. 147164 | 3 '833 |
| -2.1 | 2.782779 | 3265 | 1.287465 | 3711 |
| -2. 2 | 2.931265 | 3203 | 1.431479 | 3602 |
| -2.3 | 3.082955 | 3151 | 1. 579097 | 3503 |
| -2.4 | 3.237796 | 3097 | 1.730219 | 3409 |
| -2. 5 | 3.395734 | 3044 | 1.884752 | 3326 |

Note: Where fourth and higher differences are not negligible, their effect on interpolated values may be taken into account by using "modified second differences," $\delta_{m}^{2}$, instead of the usual second difference, $\delta^{2}$. (A good discussion of this technique, known as "throwback, " can be found in Kopal, Numerical Analysis (John Wiley, New York, 1955). The appropriate second differences, when used in Everett's interpolation formula, should give results correct to within at least one or two units in the last figure tabulated.

| t | $\tan ^{-1}\left[\frac{\mathrm{v}(\mathrm{t})}{\mathrm{u}(\mathrm{t})}\right]$ | $\delta^{2}$ | $\tan ^{-1}\left[\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]$ | $\delta^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| -2. 5 | 3.395734 | $+3045$ | 1.884752 | $+3327$ |
| -2. 6 | 3.556717 | 2998 | 2.042612 | 3247 |
| -2.7 | 3.720 698 | 2950 | 2.203719 | 3175 |
| -2.8 | 3.887629 | 2904 | 2.368001 | 3105 |
| -2.9 | 4.057464 | 2860 | 2. 535388 | 3044 |
| -3.0 | 4.230159 | 2820 | 2.705819 | 2982 |
| -3. 1 | 4.405674 | 2778 | 2.879 232 | 2926 |
| -3. 2 | 4.583967 | 2739 | 3.055571 | 2875 |
| -3.3 | 4.764999 | 2701 | 3.234785 | 2825 |
| -3. 4 | 4.948732 | 2666 | 3.416824 | 2776 |
| -3. 5 | 5.135131 | 2631 | 3.601639 | 2732 |
| -3.6 | 5.324161 | 2596 | 3.789186 | 2690 |
| -3.7 | 5. 515787 | 2564 | 3.979423 | 2650 |
| -3. 8 | 5.709 977 | 2533 | 4. 172310 | 2611 |
| -3.9 | 5.906700 | 2501 | 4.367808 | 2573 |
| -4.0 | 6.105924 | 2473 | 4.565879 | 2540 |
| -4.1 | 6.307621 | 2444 | 4. 766490 | 2505 |
| -4. 2 | 6.511762 | 2416 | 4.969606 | 2473 |
| -4. 3 | 6.718319 | 2390 | 5.175195 | 2441 |
| -4.4 | 6.927266 | 2363 | 5.383225 | 2413 |
| -4. 5 | 7. 138576 | 2338 | 5.593668 | 2384 |
| -4. 6 | 7. 352224 | 2315 | 5. 806495 | 2355 |
| -4.7 | 7.568187 | 2289 | 6.021677 | 2329 |
| -4.8 | 7.786439 | 2267 | 6.239188 | 2304 |
| -4.9 | 8.006958 | 2246 | 6.459003 | 2279 |
| -5.0 | 8.229723 | 2222 | 6.681097 | 2254 |


| t | $\tan ^{-1}\left[\frac{v(t)}{u(t)}\right]$ | $\delta^{2}$ | $\tan ^{-1}\left[\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]$ | $\delta^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| -5.0 | 8.229723 | +2222 | 6.681097 | $+2254$ |
| -5.1 | 8.454710 | 2202 | 6.905445 | 2232 |
| -5.2 | 8.681899 | 2180 | 7. 132025 | 2209 |
| -5.3 | 8.911268 | 2162 | 7.360814 | 2187 |
| -5.4 | 9. 142799 | 2142 | 7.591790 | 2166 |
| -5.5 | 9. 376472 | 2121 | 7. 824932 | 2146 |
| -5.6 | 9.612266 | 2105 | 8.060220 | 2125 |
| -5.7 | 9.850165 | 2086 | 8.297633 | 2106 |
| -5.8 | 10.090150 | 2067 | 8. 537152 | 2088 |
| -5.9 | 10.332202 | 2052 | 8.778759 | 2069 |
| -6.0 | 10.576306 | 2034 | 9.022435 | 2051 |
| -6. 1 | 10.822444 | 2017 | 9.268162 | 2034 |
| -6. 2 | 11.070599 | 2003 | 9.515923 | 2017 |
| -6. 3 | 11.320757 | 1985 | 9. 765701 | 2001 |
| -6. 4 | 11.572900 | 1971 | 10.017480 | 1984 |
| -6. 5 | 11.827014 | 1955 | 10.271243 | 1969 |
| -6.6 | 12.083083 | 1942 | 10.526975 | 1954 |
| -6. 7 | 12.341094 | 1927 | 10.784661 | 1938 |
| -6.8 | 12.601032 | 1912 | 11.044285 | 1924 |
| -6.9 | 12.862882 | 1900 | 11.305833 | 1910 |
| -7.0 | 13.126 632 | 1885 | 11.569291 | 1896 |
| -7.1 | 13.392267 | 1872 | 11.834645 | 1881 |
| -7.2 | 13.659774 | 1860 | 12.101880 | 1870 |
| -7.3 | 13.929141 | 1848 | 12.370985 | 1855 |
| -7. 4 | 14.200356 | 1833 | 12.641945 | 1843 |
| -7. 5 | 14.473404 | 1823 | 12.914748 | 1830 |


| t | $\tan ^{-1}\left[\frac{\mathrm{~V}(\mathrm{t})}{\mathrm{u}(\mathrm{t})}\right]$ | $\delta^{2}$ | $\tan ^{-1}\left[\frac{v^{\prime}(t)}{u^{\prime}(t)}\right]$ | $\delta^{2}$ |
| :---: | :---: | :---: | :---: | :---: |
| -7. 5 | 14.473404 | + 1823 | 12.914748 | $+1830$ |
| -7.6 | 14.748275 | 1811 | 13.189381 | 1819 |
| -7.7 | 15.024957 | 1799 | 13.465833 | 1806 |
| -7.8 | 15.303438 | 1786 | 13.744091 | 1794 |
| -7.9 | 15.583705 | 1777 | 14.024143 | 1783 |
| -8.0 | 15.865749 | 1766 | 14.305978 | 1771 |
| -8.1 | 16.149559 | 1753 | 14.589584 | 1761 |
| -8. 2 | 16.435122 | 1744 | 14.874951 | 1750 |
| -8. 3 | 16.722429 | 1734 | 15.162068 | 1738 |
| -8. 4 | 17.011470 | 1722 | 15.450923 | 1729 |
| -8. 5 | 17.302233 | 1713 | 15.741507 | 1717 |
| -8.6 | 17.594709 | 1703 | 16.033808 | 1709 |
| -8.7 | 17.888888 | 1693 | 16.327818 | 1697 |
| -8.8 | 18.184760 | 1684 | 16.623525 | 1689 |
| -8.9 | 18.482316 | 1674 | 16.920921 | 1678 |
| -9.0 | 18.781546 | 1666 | 17.219995 | 1669 |
| -9.1 | 19.082442 | 1654 | 17.520738 | 1661 |
| -9.2 | 19.384992 | 1647 | 17.823142 | 1650 |
| -9.3 | 19.689189 | 1639 | 18.127196 | 1642 |
| -9.4 | 19.995025 | 1629 | 18.432892 | 1632 |
| -9. 5 | 20.302490 | 1620 | 18.740220 | 1625 |
| -9.6 | 20.611575 | 1613 | 19.049173 | 1616 |
| -9.7 | 20.922273 | 1604 | 19.359742 | 1607 |
| -9.8 | 21.234575 | 1596 | 19.671918 | 1600 |
| -9.9 | 21.548473 | 1588 | 19.985694 | 1590 |
| -10.0 | 21.863959 |  | 20.301060 |  |

## Appendix B

## Some Formulae Involving Airy Functions

The Airy functions $u(t)$ and $v(t)$ are linearly independent solutions of the differential equation [Miller, 1946; Fock, 1946]

$$
\frac{d^{2} f}{d t^{2}}-t f=0
$$

i.e.,

$$
u^{\prime \prime}(t)=t u(t) \quad \text { and }
$$

$$
v^{\prime \prime}(t)=t \quad v(t) .
$$

Infinite Series Representation:

$$
\begin{aligned}
u(t) & =\frac{\sqrt{3 \pi}}{\sqrt[3]{9} \Gamma(2 / 3)} y_{1}(t)+\frac{\sqrt{3} \pi}{\sqrt[3]{3} \Gamma(1 / 3)} y_{2}(t) \\
& =1.089 \quad 929 \quad 069 \quad y_{1}(t)+0.794 \quad 570 \quad 425 y_{2}(t) \\
v(t) & =\frac{\sqrt{\pi}}{\sqrt[3]{9} \Gamma(2 / 3)} y_{1}(t)-\frac{\sqrt{\pi}}{\sqrt[3]{3} \Gamma(1 / 3)} y_{2}(t) \\
& =0.629 \quad 270 \quad 841 y_{1}(t)-0.458 \quad 745 \quad 449 y_{2}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
& y_{1}(t)=1+\frac{1}{3!} t^{3}+\frac{(1)(4)}{6!} t^{6}+\frac{(1)(4)(7)}{9!} t^{9}+\frac{(1)(4)(7)(10)}{12!} t^{12}+\ldots \\
& y_{2}(t)=t+\frac{2}{4!} t^{4}+\frac{(2)(5)}{7!} t^{7}+\frac{(2)(5)(8)}{10!} t^{10}+\frac{(2)(5)(8)(11)}{13!} t^{13}+\ldots
\end{aligned}
$$

These are valid for all $t$.

Of course, $y_{1}(t)$ and $y_{2}(t)$ are also linearly independent solutions of the differential equation satisfied by $u(t)$ and $v(t)$. In fact, $y_{1}(t)$ and $y_{z}(t)$ are obtained when the differential equation is solved by the method of Frobenius.

Wronskian Identity:

$$
u^{\prime}(t) v(t)-u(t) v^{\prime}(t)=1
$$

## Definite Integral Representation:

$$
\begin{aligned}
& u(t)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty}\left\{\exp \left(-\frac{1}{3} x^{3}+t x\right)+\sin \left(\frac{1}{3} x^{3}+t x\right)\right\} d x \\
& v(t)=\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \cos \left(\frac{1}{3} x^{3}+t x\right) d x
\end{aligned}
$$

## Representation in Terms of Bessel Functions:

If principal values are taken

$$
u(t)=\sqrt{\frac{\pi t}{3}}\left[I_{-1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)+I_{1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right]
$$

$$
v(t)=\frac{\sqrt{\pi t}}{3}\left[I_{-1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)-I_{l / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right]=\sqrt{\frac{t}{3 \pi}} K_{1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)
$$

$$
u(-t)=\sqrt{\frac{\pi t}{3}}\left[J_{-1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)-J_{l / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right]
$$

$$
\begin{aligned}
& v(-t)=\frac{\sqrt{\pi t}}{3}\left[J_{-1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)+J_{1 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right] \\
& u^{\prime}(t)=\sqrt{\frac{\pi}{3}} t\left[I_{2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)+I_{-2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right] \\
& v^{\prime}(t)=\frac{\sqrt{\pi}}{3} t\left[I_{2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)-I_{-2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right] \\
& u^{\prime}(-t)=\sqrt{\frac{\pi}{3}} t\left[J_{2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)+J_{-2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right] \\
& v^{\prime}(-t)=\frac{\sqrt{\pi}}{3} t\left[J_{2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)-J_{-2 / 3}\left(\frac{2}{3} t^{3 / 2}\right)\right]
\end{aligned}
$$

Asymptotic Expansions:
Large positive numbers: $\left[|t| \rightarrow \infty,|\arg t|<\frac{\pi}{3}\right]$

$$
\begin{aligned}
& u(t) \approx \frac{1}{t^{1 / 4}} \exp \left(\frac{2}{3} t^{3 / 2}\right) \\
& v(t) \approx \frac{1}{2 t^{1 / 4}} \exp \left(-\frac{2}{3} t^{3 / 2}\right) \\
& u^{\prime}(t) \approx t^{1 / 4} \exp \left(\frac{2}{3} t^{3 / 2}\right) \\
& v^{\prime}(t) \approx-\frac{t^{1 / 4}}{2} \exp \left(-\frac{2}{3} t^{3 / 2}\right)
\end{aligned}
$$

Large Negative Numbers: $\left[|t| \rightarrow \infty, \arg (-t)<\frac{2 \pi}{3}\right]$

$$
\begin{aligned}
& u(t) \approx \frac{1}{(-t)^{l / 4}} \cos \left[\frac{2}{3}(-t)^{3 / 2}+\frac{\pi}{4}\right], \\
& v(t) \approx \frac{1}{(-t)^{l / 4}} \sin \left[\frac{2}{3}(-t)^{3 / 2}+\frac{\pi}{4}\right], \\
& u^{\prime}(t) \approx(-t)^{l / 4} \sin \left[\frac{2}{3}(-t)^{3 / 2}+\frac{\pi}{4}\right], \\
& v^{\prime}(t) \approx-(-t)^{1 / 4} \cos \left[\frac{2}{3}(-t)^{3 / 2}+\frac{\pi}{4}\right],
\end{aligned}
$$

In general,

$$
\begin{aligned}
& \mathrm{w}_{1}(\mathrm{t})=\mathrm{u}(\mathrm{t})-\mathrm{i}-\mathrm{v}(\mathrm{t}) \\
& \mathrm{w}_{2}(\mathrm{t})=\mathrm{u}(\mathrm{t})+\mathrm{iv}(\mathrm{t}) \quad \dagger
\end{aligned}
$$

## Contour Integral Representation:

$w_{1}(t)=\frac{1}{\sqrt{\pi}} \int_{C_{1}} \exp \left(t z-\frac{1}{3} z^{3}\right) d z$
$w_{1}^{\prime}(t)=\frac{1}{\sqrt{\pi}} \int_{C_{1}} z \exp \left(t z-\frac{1}{3} z^{3}\right) d z$


It should be noted here that V.A. Fock [1946] defines $w_{1}(t)=u(t)+i v(t)$ and $w_{2}(t)=u(t)-i v(t)$.
$w_{2}(t)=\frac{1}{\sqrt{\pi}} \int_{C_{2}} \exp \left(t z-\frac{1}{3} z^{3}\right) d z$
$w_{2}^{\prime}(t)=\frac{1}{\sqrt{\pi}} \int_{C_{2}} z \exp \left(t z-\frac{1}{3} z^{3}\right) d z$


Wronskian Identity:

$$
w_{1}^{\prime}(t) w_{2}(t)-w_{1}(t) w_{2}^{8}(t)=i 2
$$

Representation in Terms of Hankel Functions:

$$
\begin{aligned}
& w_{1}(t)=e^{-i 2 \pi / 3} \sqrt{-\frac{\pi t}{3}} H_{1 / 3}^{(2)}\left[\frac{2}{3}(-t)^{3 / 2}\right] \\
& w_{2}(t)=e^{t i 2 \pi / 3 \sqrt{-\frac{\pi t}{3}} H_{1 / 3}^{(1)}\left[\frac{2}{3}(-t)^{3 / 2}\right]} \\
& w_{1}^{\gamma}(t)=e^{-i \pi / 3} \sqrt{-\frac{\pi t}{3}} H_{2 / 3}^{(2)}\left[\frac{2}{3}(-t)^{3 / 2}\right] \\
& w_{2}^{\gamma}(t)=e^{+i \pi / 3} \sqrt{-\frac{\pi t}{3}} H_{2 / 3}^{(1)}\left[\frac{2}{3}(-t)^{3 / 2}\right]
\end{aligned}
$$

## Asymptotic Forms:

$|\operatorname{targ} t|<\frac{\pi}{3}: \quad w_{\frac{1}{2}}(t) \approx \frac{1}{t} \operatorname{l/4} \exp \left(\frac{2}{3} t^{3 / 2}\right)$

$$
\mathrm{w}_{\frac{2}{\prime}}^{\prime}(\mathrm{t}) \approx \mathrm{t}^{1 / 4} \exp \left(\frac{2}{3} \mathrm{t}^{3 / 2}\right)
$$

$|t| \rightarrow \infty$,
$|\arg (-t)|<\frac{2 \pi}{3}: \quad w_{\frac{1}{2}}(t) \approx \frac{e^{\mp i \pi / 4}}{(-t)^{1 / 4}} \exp \left[\mp i \frac{2}{3}(-t)^{3 / 2}\right]$

$$
\begin{aligned}
\mathrm{w}_{\frac{1}{2}}^{q}(t) & \approx \pm i(-t)^{1 / 4} e^{\mp i \pi / 4} \exp \left[\mp i \frac{2}{3}(-t)^{3 / 2}\right] \\
& \approx \pm i(-t)^{1 / 2}{ }_{w_{1}}(t)
\end{aligned}
$$

## Values on Special Rays:

$$
\begin{array}{ll}
w_{1}\left(r e^{+i \pi / 3}\right)=e^{-i \pi / 3} w_{2}(-r) & w_{2}\left(r e^{+i \pi / 3}\right)=2 e^{+i \pi / 6} v(-r) \\
w_{1}\left(r e^{+i 2 \pi / 3}\right)=2 e^{-i \pi / 6} v(r) & w_{2}\left(r e^{+i 2 \pi / 3}\right)=e^{+i \pi / 3} w_{1}(r) \\
w_{1}\left(r e^{-i \pi / 3}\right)=2 e^{-i \pi / 6} v(-r) & w_{2}\left(r e^{-i \pi / 3}\right)=e^{+i \pi / 3} w_{1}(-r) \\
w_{1}\left(r e^{-i 2 \pi / 3}\right)=e^{-i \pi / 3} w_{2}(r) & w_{2}\left(r e^{-i 2 \pi / 3}\right)=2 e^{+i \pi / 6} v(r) \\
w_{1}\left(r e^{+i \pi / 3}\right)=e^{+i \pi / 3} w_{2}^{\prime}(-r) & w_{2}^{\prime}\left(r e^{+i \pi / 3}\right)=2 e^{-i \pi / 6} v^{\prime}(-r)
\end{array}
$$

$$
\begin{array}{ll}
w_{l}^{\prime}\left(r e^{+i 2 \pi / 3}\right)=2 e^{+i \pi / 6} v^{\prime}(r) & w_{2}^{\prime}\left(r e^{+i 2 \pi / 3}\right)=e^{-i \pi / 3} w_{l}^{\prime}(r) \\
w_{l}^{\prime}\left(r e^{-i \pi / 3}\right)=2 e^{+i \pi / 6} v^{\prime}(-r) & w_{2}^{\prime}\left(r e^{-i \pi / 3}\right)=e^{-i \pi / 3} w_{1}^{\prime}(-r) \\
w_{1}^{\prime}\left(r e^{-i 2 \pi / 3}\right)=e^{+i \pi / 3} w_{2}^{q}(r) & w_{2}^{\prime}\left(r e^{-i 2 \pi / 3}\right)=2 e^{-i \pi / 6} v^{8}(r)
\end{array}
$$

## Derivatives:

If $w(t)$ is any solution of the differential equation $\frac{d^{2} w}{d t^{2}}-t w=0$, then

$$
\begin{aligned}
& w_{w}^{(2)}(t)=t w(t) \quad\left(\text { Note that } w^{(n)}(t)=\frac{d^{n}}{d t^{n}} w(t)\right) \\
& w_{w}^{(3)}(t)=w(t)+t w^{9}(t) \\
& w_{w}^{(4)}(t)=t^{2} w(t)+2 w^{9}(t) \\
& w^{(6)}(t)=4 t w(t)+t^{2} w^{8}(t) \\
& \left.w^{(7)}(t)=9 t^{2}+4\right) w(t)+6 t w^{9}(t) \\
& w^{(8)}(t)=\left(t^{4}+28 t\right) w(t)+12 t^{2} w^{9}(t) \\
& w^{(9)}(t)=\left(16 t^{3}+28\right) w(t)+\left(t^{4}+52 t\right) w^{9}(t) \\
& w^{(10)}(t)=\left(t^{5}+100 t^{2}\right) w(t)+\left(20 t^{3}+80\right) w^{8}(t)
\end{aligned}
$$

$w^{(11)}(t)=\left(25 t^{4}+280 t\right) w(t)+\left(t^{5}+160 t^{2}\right) w^{\prime}(t)$
$w^{(2)}(0)=0$
$w^{(5)}(0)=0$
$\mathrm{w}^{(8)}(0)=0$
$w^{(3)}(0)=w(0)$
$w^{(6)}(0)=4 w(0)$
$w^{(9)}(0)=28 \mathrm{w}(0)$
$w^{(4)}(0)=2 w^{\prime}(0)$
$w^{(7)}(0)=10 w^{\prime}(0)$
$w^{(10)}(0)=80 w^{P}(0)$

A Taylor Series Expansion:
If $w(t)$ is any solution of $\frac{d^{2} w}{d^{2}}-t w=0$, then

$$
\begin{aligned}
& \mathrm{w}(\mathrm{t}+\mathrm{h})=a(\mathrm{t}, \mathrm{~h}) \mathrm{w}(\mathrm{t})+\beta(\mathrm{t}, \mathrm{~h}) \mathrm{w}^{\text {p }}(\mathrm{t}) \\
& \mathrm{w}^{\mathrm{g}}(\mathrm{t}+\mathrm{h})=a^{\mathrm{g}}(\mathrm{t}, \mathrm{~h}) \mathrm{w}(\mathrm{t})+\beta^{p}(\mathrm{t}, \mathrm{~h}) \mathrm{w}^{\prime}(\mathrm{t})
\end{aligned}
$$

where

$$
a(t, h)=1+t a_{2}(h)+a_{3}(h)+t^{2} a_{4}(h)+4 t a_{5}(h)+\left(t^{3}+4\right) a_{6}(h)
$$

$$
+9 t^{2} a_{7}(h)+\left(t^{4}+28 t\right) a_{8}(h)+\left(16 t^{3}+28\right) a_{9}(h)
$$

$$
+\left(t^{5}+100 t^{2}\right) a_{10}(h)+\left(25 t^{4}+280 t\right) a_{11}(h)+\ldots
$$

$$
\beta(t, h)=a_{1}(h)+t a_{3}(h)+2 a_{4}(h)+t^{2} a_{5}(h)+6 t a_{6}(h)+\left(t^{3}+10\right) a_{7}(h)
$$

$$
+12 t^{2} a_{8}(h)+\left(t^{4}+52 t\right) a_{9}(h)+\left(20 t^{3}+80\right) a_{10}(h)+\left(t^{5}+160 t^{2}\right) a_{11}(h)+\ldots
$$

$$
\begin{aligned}
& a^{\prime}(t, h)=t a_{1}(h)+a_{2}(h)+t^{2} a_{3}(h)+4 t a_{4}(h)+\left(t^{3}+4\right) a_{5}(h)+9 t^{2} a_{6}(h) \\
& +\left(t^{4}+28 t\right) a_{7}(h)+\left(16 t^{3}+28\right) a_{8}(h)+\left(t^{5}+100 t^{2}\right) a_{9}(h) \\
& +\left(25 t^{4}+280 t\right) a_{10}(h)+\ldots \\
& \beta^{\prime}(t, h)=1+t a_{2}(h)+2 a_{3}(h)+t^{2} a_{4}(h)+6 t a_{5}(h)+\left(t^{3}+10\right) a_{6}(h) \\
& +12 t^{2} a_{7}(h)+\left(t^{4}+52 t\right) a_{8}(h)+\left(20 t^{3}+80\right) a_{9}(h)+\left(t^{5}+160+t^{2}\right) a_{10}(h)+\ldots \\
& \text { and } \\
& a_{k}(h)=\frac{h^{k}}{k!} \quad(k=0,1,2, \ldots) . \\
& \text { Note that } \\
& a_{k}(h)=\frac{h}{k} a_{k-1}(k=1,2,3, \ldots) \quad .
\end{aligned}
$$

Formulas Involving Complex Arguments:
Let $z=x+i y$ where $x, y$ are real. If $w(z)$ is any solution of the differential equation

$$
\frac{d^{2} w}{d z^{2}}-z w=0
$$

the following expressions are fairly convenient for computation when $y$ is small and $x$ is not too large:

$$
w(z)=\left[\theta(x, y) w(x)+\phi(x, y) w^{\prime}(x)\right]+i\left[\xi(x, y) w(x)+\eta(x, y) w^{\gamma}(x)\right]
$$

where

$$
\begin{aligned}
& \theta(x, y)=1-\frac{x y^{2}}{2}+\frac{x^{2} y^{4}}{24}-\frac{\left(x^{3}+4\right) y^{6}}{720}+\frac{\left(x^{4}+28 x\right) y^{8}}{40,320}-\frac{\left(x^{5}+100 x^{2}\right) y^{10}}{3,628,800}+\ldots \\
& \phi(x, y)=\frac{y^{4}}{12}-\frac{x y^{6}}{120}+\frac{x^{2} y^{8}}{3360}-\frac{\left(x^{3}+4\right) y^{10}}{181,440}+\ldots \\
& \xi(x, y)=-\frac{y^{3}}{6}+\frac{x y^{5}}{30}-\frac{x^{2} y^{7}}{560}+\frac{\left(4 x^{3}+7\right) y^{9}}{90,720}-\frac{\left(5 x^{4}+56 x\right) y^{11}}{7,983,360}+\ldots \\
& \eta(x, y)=y-\frac{x y^{9}}{6}+\frac{x^{2} y^{5}}{120}-\frac{\left(x^{3}+10\right) y^{7}}{5040}+\frac{\left(x^{4}+52 x\right) y^{9}}{362,880}-\frac{\left(x^{5}+160 x^{2}\right) y^{11}}{39,916,800}+\ldots
\end{aligned}
$$

and

$$
w^{\prime}(z)=\left[\theta^{\prime}(x, y) w(x)+\phi^{\prime}(x, y) w^{\prime}(x)\right]+i\left[\xi^{\prime}(x, y) w(x)+\eta^{\prime}(x, y) w^{\prime}(x)\right]
$$

where

$$
\begin{aligned}
& \theta^{\prime}(x, y)=-\frac{y^{2}}{2}+\frac{x y^{4}}{6}-\frac{x^{2} y^{6}}{80}+\frac{\left(4 x^{3}+7\right) y^{8}}{10,080}-\frac{\left(5 x^{4}+56 x\right) y^{10}}{725,760}+\ldots \\
& \phi^{\prime}(x, y)=1-\frac{x y^{2}}{2}+\frac{x^{2} y^{4}}{24}-\frac{\left(x^{3}+10\right) y^{6}}{720}+\frac{\left(x^{4}+52 x\right) y^{8}}{40,320}-\frac{\left(x^{5}+160 x^{2}\right) y^{20}}{3,628,800}+\ldots \\
& \xi^{\prime}(x, y)=x y-\frac{x^{2} y^{3}}{6}+\frac{\left(x^{3}+4\right) y^{5}}{120}-\frac{\left(x^{4}+28 x\right) y^{7}}{5040}+\frac{\left(x^{5}+100 x^{2}\right) y^{9}}{362,880}-\ldots \\
& \eta^{8}(x, y)=-\frac{y^{3}}{3}+\frac{x y^{5}}{20}-\frac{x^{2} y^{7}}{420}+\frac{\left(x^{3}+4\right) y^{9}}{18,144}-\ldots
\end{aligned}
$$

Note: An interesting discussion of Airy functions and relations among the many differing notations is found in an unpublished report (dated Dec. 1959) by Nelson A. Logan of the Lockheed Aircraft Co., Sunnyvale, Calif.

Appendix. C
A Note on the Conductivity of the Lower Ionosphere at VLF
In the theory of ionospheric propagation of radio waves, it is nearly always assumed that the collision frequency of electrons with neutral particles is independent of electron energy. In fact, the Appleton-Hartree equations were developed on this basis. It has been suggested recently [Alpert et al. 1953; Sen and Wyller, 1960; Phelps, 1960] that the calculation of the propagation constant for a weakly ionized medium should take into account this energy dependence. While the theory has been so generalized by Sen and $W$ yller [1960] and Johler and Harper [1962], it seems worthwhile to present a somewhat simplified account of the consequences of a linear dependence of the collision frequency on electron energy. Furthermore, this sheds some light on the validity of describing the electrical characteristics of the lower ionosphere in terms of a conductivity.
W.P. Allis [1956] and others have related the components of the dielectric tensor to integrals involving the angular frequency $\omega$, the gyro frequency of electrons $\omega_{H}$, the electron plasma frequency, the electron density $N$, the normalized electron energy distribution function $f_{0}$, the frequency of mementum transfer collisions of electrons with gas molecules $v(u)$. When a constant and uniform magnetic field is applied in the $z$ direction the dielectric constant of the ionized medium is a tensor of the form

$$
(\varepsilon)=\left(\begin{array}{ccc}
\varepsilon^{\prime} & -\mathrm{iq} & 0 \\
\mathrm{iq} & \varepsilon^{\prime} & 0 \\
0 & 0 & 0
\end{array}\right)
$$

where

$$
\begin{aligned}
& \varepsilon^{\prime}=\frac{1}{2}\left(\epsilon_{L}+\epsilon_{R}\right) \\
& q=\frac{1}{2}\left(\varepsilon_{L}-\epsilon_{R}\right) \\
& \varepsilon^{\prime \prime}=\varepsilon_{P} .
\end{aligned}
$$

In this, the dielectric constants $\varepsilon_{L}, \varepsilon_{R}$ and $\varepsilon_{P}$ may be written in the form

$$
\begin{aligned}
\epsilon_{L} & =\varepsilon\left(\omega+\omega_{H}\right) \\
{ }^{\epsilon_{R}} & =\epsilon\left(\omega-\omega_{H}\right) \\
\varepsilon_{P} & =\varepsilon(\omega)
\end{aligned}
$$

where, according to Molmud [1959],

$$
i \varepsilon(\Omega) \omega=\sigma(\Omega)=-\frac{4 \pi}{3} \varepsilon_{0} \omega_{0}^{2} \int_{0}^{\infty} \frac{u^{3 / 2}}{v(u)+i \Omega} \frac{\partial f_{o}}{\partial u} d u
$$

In the above, $\sigma(\Omega)$ is a generalized conductivity which is a function of a generalized frequency $\Omega$.

In the case where the electrons are in thermal equilibrium with the gas, the energy distribution of the electrons is Maxwellian and given by

$$
f_{o}=(e / \pi k T)^{3 / 2} e^{-e u / k T}
$$

In the case of weakly ionized dry air, the collision frequency $v(u)$ is now believed to be approximately proportional to the electron energy u. In fact, Phelps [1960] has shown that, for low energy electrons in nitrogen,

$$
v(u) \cong 1.2 \times 10^{-7} \mathrm{~N}\left(\mathrm{~N}_{2}\right) \mathrm{u} \mathrm{sec}^{-1}
$$

where $u$ is in electron-volts/c.c. For present purposes, we will just set $v(u)=a u$ where $a$ is a constant. Furthermore, a normalized collision frequency $\nu_{1}$ is chosen such that

$$
v_{1}=\mathrm{akT} / \mathrm{e}
$$

where $k$ is the usual Boltzmann constant, $T$ is the absolute temperature in degrees Kelvin and $e$ is the electronic charge.

Using the simplifications described in the preceding paragraph, it is seen that

$$
\sigma(\Omega)=\frac{4 \pi}{3} \varepsilon_{0} \omega_{0}^{2}\left(\frac{a}{\pi \nu_{1}}\right)^{3 / 2} \int_{0}^{\infty} \frac{u^{3 / 2}}{a u+i \Omega} e^{-u a / \nu_{1}} d\left(\frac{u a}{v_{1}}\right)
$$

This is essentially the formulas given by Phelps [1960], Sen and Wyller [1960] and others. As they have indicated, the integral can be expressed in terms of the "semi-conductor integral" $E_{p}(x)$ defined by

$$
E_{p}(x)=\frac{1}{(p!)} \int_{0}^{\infty} a^{p}\left(a^{2}+x^{2}\right)^{-1} e^{-\alpha} d \alpha
$$

These have been tabulated by Dingle, Arndt and Roy [1956] for integral and half-integral values of $p$ in the range $-1 / 2$ to +5 . It easily follows that

$$
\sigma(\Omega)=\frac{\varepsilon_{0} \omega_{0}^{2}}{v_{1}}\left[\frac{5}{2} \mathrm{E}_{5 / 2}(\mathrm{x})-\mathrm{i} \times \mathrm{E}_{3 / 2}(\mathrm{x})\right]
$$

where $\mathrm{x}=\Omega / v_{1}$.
If, on the other hand, $v(u)$ had been replaced by a constant $v_{o}$, as is conventionally done, we would have arrived at the standard result

$$
\sigma(\Omega)=\frac{\epsilon_{0} \omega_{0}^{2}}{v_{0}+i \Omega}
$$

This can be written in the form

$$
\sigma(\Omega)=\epsilon_{\mathrm{o}} \omega_{\mathrm{r}} \frac{1}{1+\mathrm{i} \beta}
$$

where $\beta=\frac{\Omega}{\nu_{0}} \quad$ and $\quad \omega_{r}=\frac{\omega_{o}^{2}}{\nu_{0}}$.

The parameter $\omega_{r}$ occurs often in the theory of VLF propagation which is usually formulated on the assumption of an energy independent collision frequency. Furthermore, at VLF $\omega \ll \omega_{H}$ so that

$$
\beta \cong \pm \frac{\omega_{\mathrm{H}}}{v_{\mathrm{o}}}
$$

The parameter $\omega_{H} / \nu_{o}$ also occurs consistently in the presentation of theoretical results. Thus, it appears that a convenient way to illustrate the implications of the energy dependent collision frequency is to define effective values $\left(\omega_{r}\right)$ and $\beta_{e}$ as follows

$$
\sigma(\Omega)=\epsilon_{\mathrm{o}}\left(\omega_{\mathrm{r}}\right) \frac{1}{1+\mathrm{i} \beta_{\mathrm{e}}}
$$

Thus

$$
\left(\omega_{\mathrm{r}}\right)=\omega_{\mathrm{r}_{1}} a_{\mathrm{e}}(\mathrm{x})
$$

where

$$
\begin{gathered}
\omega_{r_{1}}=\frac{\omega_{0}^{2}}{\nu_{1}} \\
a_{e}(x)=\frac{\frac{5}{2} E_{3 / 2}(x)}{\left[\frac{5}{2} E_{5 / 2}(x)\right]^{2}+\left[x E_{3 / 2}(x)\right]^{2}}
\end{gathered}
$$

and

Also, it is seen that

$$
\beta_{e}(x)=\frac{x E_{3 / 2}(x)}{\frac{5}{2} E_{5 / 2}(x)}
$$

where, as above, $x=\frac{\Omega}{\nu_{1}}$.

Using the numerical values of $E_{3 / 2}(x)$ and $E_{5 / 2}(x)$ given by Dingle, Arndt and Roy [1956], the values of $a_{e}(x)$ and $\beta_{e}(x)$ are plotted as a function of x and given in Table C-1. It is seen that for small values of $\mathrm{x}, \alpha(\mathrm{x})$ asymptotically approaches the constant value of $2 / 3$, whereas $\beta_{e}(x)$ is asymptotically approaching the value $2 x$. On the other hand, for large values of $x$, the respective asymptotes are 0.4 and $\mathrm{x} / 2.5$.

If the collision frequency was chosen to be independent of energy, the corresponding values $a_{\mathrm{e}}(\mathrm{x})$ and $\beta_{\mathrm{e}}(\mathrm{x})$ would be simply 1.0 and x , respectively. It is thus concluded that a linear energy dependence for the collision frequency is not going to lead to any essential modifications to the theory of VLF propagation. In fact, most of the numerical results on the characteristics of the VLF modes can be adapted directly to the energy dependent case if $\omega_{r}$ and the ratio $\omega_{\mathrm{H}} / \nu$ are given their more general meaning.

Table C-1

| $x$ | $a_{e}(x)$ | $\beta_{e}(x)$ |
| :---: | :--- | :--- |
| 0.01 | 0.66546 | 0.01755 |
| 0.02 | 0.66363 | 0.03322 |
| 0.03 | 0.66164 | 0.04776 |
| 0.05 | 0.65738 | 0.07451 |
| 0.1 | 0.64665 | 0.13256 |
| 0.2 | 0.62691 | 0.22772 |
| 0.4 | 0.59561 | 0.37771 |
| 0.6 | 0.57184 | 0.50127 |
| 0.8 | 0.55300 | 0.61080 |
| 1.0 | 0.53771 | 0.71148 |
| 1.2 | 0.52487 | 0.80578 |
| 1.6 | 0.50441 | 0.98308 |
| 2.0 | 0.48921 | 1.15113 |
| 2.5 | 0.47428 | 1.35171 |
| 3.0 | 0.46293 | 1.54763 |
| 3.5 | 0.45416 | 1.74084 |
| 4.0 | 0.44701 | 1.93203 |
| 5.0 | 0.43617 | 2.31099 |
| 6.0 | 0.42862 | 2.69037 |
| 7.0 | 0.42315 | 3.07048 |
| 8.0 | 0.41911 | 3.45146 |
| 9.0 | 0.41579 | 3.83384 |
| 10.0 | 0.41360 | 4.22111 |
| 11.0 | 0.41177 | 4.60788 |
| 12.0 | 0.41011 | 4.99500 |
| 13.0 | 0.40893 | 5.38480 |
| 14.0 | 0.40783 | 5.77448 |
| 15.0 | 0.40679 | 6.16362 |
| 16.0 | 0.40626 | 6.55735 |
| 17.0 | 0.40545 | 6.94835 |
| 20.0 | 0.40395 | 8.12884 |
|  |  |  |
|  |  |  |
|  |  |  |

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Cryogenic Engineering. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Cryogeric Technical Services.
Ionosphere Research and Propagation. Low Frequency and Very Low Frequency Research. lonosphcre Research. Prediction Services. Sun-Earth Relationships. Field Fngineering. Radio Warning Services.
Radio Propagation Engineering. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.
Radio Standards. High Frequency Electrical Standards. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time lnterval Standards. Electronic Calibration Center. Millimeter-Fave Research. Microwave Circuit Standards.
Radio Systems. High Frequency and Very High Frequency Research. Modulation Research. Antenna Research. Navigation Systems.
Upper Atmosphere and Space Physics. Upper Atmosphere and Plasma Physics. lonosphere and Exosphere Scatter. Airglow and Aurora. lonospheric Radio Astronomy.

NBS


[^0]:    In appendix $A$, the se functions are tabulated to six decimal places at intervals of 0.1 from $t=-10.0$ to $t=+5.0$.

